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UV finiteness of Pohlmeyer-reduced form of the $AdS_5 \times S^5$ superstring theory

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Abstract: We consider the Pohlmeyer-type reduced theory found by explicitly solving the Virasoro constraints in the formulation of $AdS_5 \times S^5$ superstring in terms of supercoset currents. The resulting set of classically equivalent, integrable Lagrangian equations of motion has the advantage of involving only a physical number of degrees of freedom and yet being 2d Lorentz invariant. The corresponding reduced theory action may be written as a gauged WZW model coupled to fermions with further bosonic and fermionic potential terms. Since the $AdS_5 \times S^5$ superstring sigma model is conformally invariant, its classical relation to the reduced theory may extend to the quantum level only if the latter is, in fact, UV finite. This theory is power counting renormalizable with the only possible divergences being of potential type. We explicitly verify its 1-loop finiteness and show that the 2-loop divergences are, in general, scheme dependent and vanish in dimensional reduction scheme. We expect that the reduced theory is finite to all orders in the loop expansion.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, Sigma Models

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Contents

1 Introduction

Recent remarkable progress in understanding the spectrum of states with large quantum numbers in $AdS_5 \times S^5$ string theory or dual $\mathcal{N} = 4$ SYM theory was achieved via interplay of various perturbative data from gauge theory and string theory linked together by the assumption of exact integrability. It remains an outstanding problem to derive the corresponding asymptotic Bethe ansatz equations directly from first principles — from quantum superstring theory. That would be facilitated if the corresponding integrable $AdS_5 \times S^5$ sigma model admitted a formulation in terms of elementary excitations with two-dimensional Lorentz covariant S-matrix. Such a formulation may also make more straightforward the generalization of the asymptotic Bethe Ansatz to the case when both strings and dual operators have finite length, i.e. to the case of closed strings on the cylinder $R_t \times S^1$.

With this motivation in mind here we shall continue the study of the Pohlmeyer-reduced [1] formulation of gauge-fixed $AdS_5 \times S^5$ superstring [2–4]. This theory (which we shall refer to as the "reduced theory") is a generalized sine-Gordon or non-abelian Toda type two-dimensional Lorentz-invariant sigma model which is closely related to the original

Green-Schwarz (GS) superstring sigma model [5]. It is constructed by writing the GS superstring equations of motion in terms of the components of the $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$ supercoset current, fixing the conformal and κ-symmetry gauges and then reconstructing the action that reproduces the equation of motion for the remaining physical number of degrees of freedom.

While the resulting reduced theory is classically equivalent to the original $AdS_5 \times S^5$ GS superstring (and, in particular, it is also classically integrable) it is a priori unclear if the corresponding quantum theories should be closely related. In general, the classical Pohlmeyer reduction assumes two-dimensional conformal invariance but for sigma models with target spaces involving S^n or AdS_n factors (and no bosonic WZ couplings) that symmetry may hold also at the quantum level only in very exceptional cases like the $AdS_5 \times S^5$ GS superstring. The minimal consistency requirement for the conjecture that the classical equivalence between the GS superstring and the reduced theory may extend to the quantum level is then the finiteness of the reduced theory — the cancellation of the UV divergences in world-sheet perturbation theory. This means the absence of any new dynamically generated scale in addition to the classical mass parameter in the potential introduced in the process of fixing the classical conformal diffeomorphism symmetry (this procedure spontaneously breaks the underlying conformal symmetry of the GS superstring in conformal gauge while preserving two-dimensional Lorentz invariance).

Our aim below will be to demonstrate the cancellation of the 1-loop and 2-loop divergences in the reduced theory which also gives a strong indication of all-loop finiteness.

Let us first briefly discuss what is known about the $AdS_5 \times S^5$ superstring theory. The classical theory [5] generalizes the $AdS_5 \times S^5$ bosonic sigma model to the presence of GS fermions incorporating self-dual 5-form coupling. The potential importance of integrability of this model (motivated by the known integrability of its bosonic part) was recognized early on [6, 7]; the classical intergrability was proved in the full theory including fermions in [8] (see also [9, 10]; for a review see [11]). Given the global symmetry, uniqueness of the (2-derivative) action and analogy with WZW theory the action is expected to be UV finite to all orders [5] and that was directly verified at 1-loop [12, 13] and 2-loop [14] orders. The classical integrability appears to extend to the quantum level as is effectively verified by the matching of the 1-loop [13] and 2-loop [14] corrections to spinning string energies to the strong-coupling predictions of the asymptotic Bethe Ansatz (see, e.g., [15] and [16]).[1]

The GS action has a well-known peculiarity in that to carry out its perturbative expansion it is necessary to choose a non-trivial background for the closed string coordinates and expand around it. The background introduces a fiducial mass scale (spontaneously breaking two-dimensional conformal invariance) and also spontaneously breaks the two-dimensional Lorentz invariance at the level of interaction terms in the action. That happens, for example, when one expands near a null geodesic or uses a version of light-cone (l.c.) gauge in $AdS_5 \times S^5$ [6, 20, 21]. While this step is a natural one when computing quantum superstring corrections to specific string states, it is a complication in general considerations (e.g., in computing the underlying factorized S-matrix). In particular, the l.c. gauge fixed

[1] Quantum integrability was also argued for in the closely related pure spinor formulation of $AdS_5 \times S^5$ superstring [17–19].

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$AdS_5 \times S^5$ GS superstring action has a complicated interaction structure making the direct computation of the corresponding magnon-type or BMN excitation S-matrix problematic beyond the tree level [10]. Another complication is that when formally expanded near a particular background the GS action is not power-counting renormalizable [14, 22] and one is to rely on a judicious choice of regularization (and measure) to verify the cancellation of the UV divergences.

Remarkably, these problems are absent in the quantum theory as defined in terms of the supercoset current variables, i.e. defined by the reduced theory action [2]. The corresponding fermionic kinetic terms have standard two-dimensional Dirac form and thus the two-dimensional Lorentz covariant fermionic propagators are defined without independently of a bosonic string coordinate background. Moreover, the reduced theory action is power counting renormalizable and relatively straightforward to quantize, as its structure is similar to that of two-dimensional supersymmetric gauged G/H WZW model supplemented with a bosonic potential and a "Yukawa" interaction term.[2]

The quadratic part of the reduced theory action has the same form as that of the GS superstring expanded near the BMN vacuum, i.e. as the GS action in maximally-supersymmetric plane wave background in the l.c. gauge [23, 24]: eight two-dimensional scalars together with eight two-dimensional Majorana fermions, all with equal mass μ. The interaction terms differ, but one may hope that there exists a certain transformation relating the corresponding S-matrices.[3] Since both the $AdS_5 \times S^5$ superstring and the corresponding reduced theory are expected to be conformal theories, the parameter μ should be the only scale on which the quantum S-matrices should depend. While the S-matrix corresponding the BMN vacuum is not two-dimensional Lorentz invariant, the one appearing in the reduced theory should be Lorentz invariant (i.e. the 4-point scattering matrix should depend only on the difference of the two rapidities). This puts the reduced theory into the same class of integrable theories as the solvable $O(n)$ sigma models.

This motivates the study of the reduced theory at the quantum level even regardless its relation to the quantum GS superstring theory: it appears to be a remarkable finite integrable model with several unique features.

Below in section 2 we shall start with a review of the reduced theory action using an explicit parametrization of the fermionic variables and clarifying on the way several important features of this theory. As was already mentioned, the construction of Pohlmeyer-reduced theory (see [2] and also [25] and references therein) involves several steps:

(i) start with the GS equations (and the Maurer-Cartan equations) written in terms of the components of the $\frac{\widehat{F}}{G} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$ supercoset current;

(ii) solve the conformal gauge constraints introducing a new set of field variables directly (algebraically) related to the currents, fixing the residual conformal diffeomorpisms

[2] As we shall see, the "Yukawa" interaction is effectively responsible for the UV finiteness of the "gWZW+potential" model.

[3] These are expected to have closely related symmetries: $\mathrm{PSU}(2|2) \times \mathrm{PSU}(2|2)$ in the GS superstring case [21, 26] and $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ in the reduced theory case [2] — the latter is formally the same as the bosonic part of the former but their precise relation needs to be clarified further.

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and κ-symmetry gauge in the process;

(iii) reconstruct an action for the remaining field equations in terms of the new (physical) variables.

The resulting reduced theory action defines a massive integrable two-dimensional field theory. Its construction thus involves a non-local map between the original coset coordinate fields and current variables that preserves the integrable structure and allows the reconstruction of the classical solutions of the GS superstring action from classical solutions of reduced theory action, i.e. the solitonic solutions in the two models are in direct correspondence.[4]

The bosonic fields of the reduced theory are $g \in G = \mathrm{Sp}(2,2) \times \mathrm{Sp}(4) \subset \mathrm{PSU}(2,2|4)$ and the two-dimensional gauge field A_μ taking values in the algebra of $H = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \subset G$. In addition, there are fermionic fields $\Psi_{_R}, \Psi_{_L}$ (directly related to fermionic currents of the GS superstring) which are two-dimensional Majorana spinors with the standard kinetic terms transforming under both $\mathrm{Sp}(2,2)$ and $\mathrm{Sp}(4)$ and thus linking together the two sets of bosons (corresponding effectively to the "transverse" string fluctuations in AdS_5 and S^5).[5] In the special case when $AdS_5 \times S^5$ is replaced by $AdS_2 \times S^2$ the corresponding reduced theory is equivalent [2] to the $\mathcal{N} = 2$ super sine-Gordon model (there H is trivial).

At the level of the equations of motion of the reduced theory it is possible to fix the $A_\mu = 0$ gauge; the equations then become equivalent to a fermionic generalization of non-abelian Toda equations. The linearization of the equations of motion in the gauge $A_\mu = 0$ around the trivial vacuum $g = \mathbb{1}$ gives 8+8 bosonic and fermionic degrees of freedom with mass μ and suggests that the symmetry of resulting relativistic S-matrix should be $H = [\mathrm{SU}(2)]^4$.

The potential term is multiplied by the "built-in" classical scale parameter μ which is a remnant of gauge-fixing the conformal diffeomorphisms at the classical level. Consistency then requires that the reduced theory be also UV finite, i.e. while a priori the μ-dependent terms in the reduced theory action may renormalize, the fermions should cancel the bosonic renormalization.

This is indeed what happens in the $AdS_2 \times S^2$ case (i.e. in the $\mathcal{N} = 2$ super sine-Gordon model). As we shall see in section 4 below, this is also true in the general $AdS_5 \times S^5$ case: we shall demonstrate the cancellation of UV divergences at the 1-loop and 2-loop orders in the natural dimensional reduction regularization scheme.[6] We believe that similar cancellations should extend to all orders in perturbation theory. Then the theory is UV finite and μ remains an arbitrary conformal symmetry gauge fixing parameter at the quantum level.

[4]This correspondence was used in [27].

[5] This model is kind of "hybrid" of a WZW model based on a supercoset (where fermions are in "off-diagonal" blocks of a supermatrix field but have non-unitary second-derivative kinetic terms) and a two-dimensional supersymmetric version of a G/H gWZW model where fermions have the standard first-order kinetic terms but take values in the coset part of the algebra of the group G.

[6]The same scheme was used in [14] where the 2-loop finiteness of the $AdS_5 \times S^5$ GS superstring was verified.

The cancellation of divergences is presumably related to a hidden symmetry that should have its origin in κ-symmetry of the original GS action that relates the coefficients of the "kinetic" and the WZ terms in the action (which, under the reduction, become the potential and the Yukawa terms in the reduced action).

There are several conceptual issues that remain to be clarified before one would be able to claim that the quantum reduced theory is indeed directly relevant for solving the quantum $AdS_5 \times S^5$ superstring theory. These include the precise mapping between observables and conserved charges (cf. [4]) and understanding the relation between massive S-matrices computed by expanding near the respective vacua. The ultimate motivation for the study of the reduced theory is the hope that it may be more straightforward to define as a quantum integrable theory and thus easier to solve than the original $AdS_5 \times S^5$ GS superstring model. To demonstrate this remains a program for the future.

2 Reduced theory for $AdS_5 \times S^5$ superstring

In this section we shall review the structure of the reduced theory action.

Our starting point is the $AdS_5 \times S^5$ superstring action [5] written in terms of currents for the supercoset[7]

$$\frac{\widehat{F}}{G} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(2,2) \times \mathrm{Sp}(4)}$$

The currents take values in the superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$ which is a quotient of $su(2,2|4)$ by elements proportional to unit matrix.

Let us first discuss the explicit parametrization of the corresponding supermatrices.

2.1 Supercoset parametrization, currents and gauge fixing

An element of $su(2,2|4)$ can be written as an 8×8 matrix

$$M = \begin{pmatrix} A & X \\ X^\dagger \Sigma & B \end{pmatrix}, \qquad \mathrm{Str}\, M = \mathrm{tr} A - \mathrm{tr} B = 0\,, \qquad A \in u(2,2)\,, \qquad B \in u(4). \tag{2.1}$$

Let us define the 4×4 matrices Σ and K (we follow the notation of [2, 11, 28]; I denotes a unit matrix of an appropriate dimension)

$$\Sigma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad K = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad [\Sigma, K] = 0, \ \ \Sigma^2 = I, \ \ K^2 = -I \tag{2.2}$$

The superalgebra $su(2,2|4)$ admits a Z_4 automorphism [29], i.e. its elements can be split into four orthogonal subspaces $\widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_1 \oplus \widehat{\mathfrak{f}}_2 \oplus \widehat{\mathfrak{f}}_3$, with $[\widehat{\mathfrak{f}}_i, \widehat{\mathfrak{f}}_j] = \widehat{\mathfrak{f}}_{i+j \ (mod\ 4)}$ in the following way:

$$M_{0,2} = \begin{pmatrix} A_{0,2} & 0 \\ 0 & B_{0,2} \end{pmatrix}, \quad A_{0,2} = \frac{1}{2}(A \pm K A^t K)\,, \qquad B_{0,2} = \frac{1}{2}(B \pm K B^t K), \tag{2.3}$$

[7] The bosonic part of the PSU(2,2|4) group is SU(2,2) × SU(4) or SO(2,4) × SO(6) and an equivalent form of the subgroup is SO(1,4) × SO(5).

$$M_{1,3} = \begin{pmatrix} 0 & X_{1,3} \\ X_{1,3}^\dagger \Sigma & 0 \end{pmatrix}, \quad X_{1,3} = \frac{1}{2}[X \pm iK(X^\dagger \Sigma)^t K] = \frac{1}{2}(X \pm i\Sigma K X^* K). \tag{2.4}$$

Here $A_0 \in sp(4)$, $B_0 \in sp(4)$, i.e. M_0 belongs to $sp(2,2) \oplus sp(4)$, while M_2 is in the bosonic part of the coset subspace of the algebra. M_1 and M_3 are expressed in terms of the real and imaginary parts of the complex 4×4 matrix X. This split is a "reality decomposition" implemented by the projectors applied to X:

$$X_{1,3} = \mathcal{P}_\pm X \equiv \frac{1}{2}(X \pm i\Sigma K X^* K), \qquad \mathcal{P}_\pm^2 = \mathcal{P}_\pm . \tag{2.5}$$

Thus the elements from $\widehat{\mathfrak{f}}_1$ and $\widehat{\mathfrak{f}}_3$ should satisfy the following conditions

$$X_1^* = -i\Sigma K X_1 K, \qquad X_3^* = i\Sigma K X_3 K, \tag{2.6}$$

which can be solved explicitly in terms of 4×4 matrices $\mathcal{X}_{1,3}$ with independent *real* Grassmann elements

$$X_1 = \mathcal{X}_1 + i\Sigma K \mathcal{X}_1 K, \qquad X_3 = \mathcal{X}_3 - i\Sigma K \mathcal{X}_3 K . \tag{2.7}$$

The $AdS_5 \times S^5$ GS action [5, 29, 31] is constructed by starting with an element f of $\widehat{F} = \mathrm{PSU}(2,2|4)$, defining the current $\mathrm{J} = f^{-1}df$ and then splitting the current according to the Z_4 decomposition of $\widehat{\mathfrak{f}}$ $(\mu, \nu = (0,1))$

$$\mathrm{J}_\mu = f^{-1}\partial_\mu f = \mathcal{A}_\mu + Q_{1\mu} + P_\mu + Q_{2\mu}, \qquad \mathcal{A} \in \widehat{\mathfrak{f}}_0, \quad Q_1 \in \widehat{\mathfrak{f}}_1, \quad P \in \widehat{\mathfrak{f}}_2, \quad Q_2 \in \widehat{\mathfrak{f}}_3 . \tag{2.8}$$

Here $\mathcal{A}_\mu$ belongs to the algebra of the subgroup G defining the $\widehat{F}/G$ coset, i.e. $G = \mathrm{Sp}(2,2) \times \mathrm{Sp}(4)$ (isomorphic to $\mathrm{SO}(1,4) \times \mathrm{SO}(5)$), P is in the bosonic coset (i.e. $AdS_5 \times S^5$) component, and Q_1, Q_2 are the fermionic currents. The Lagrangian in conformal gauge is

$$L = \mathrm{Str}\left[P_+ P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})\right], \tag{2.9}$$

which should be supplemented by the Virasoro (conformal gauge) constraints

$$\mathrm{Str}(P_+ P_+) = 0, \qquad \mathrm{Str}(P_- P_-) = 0 . \tag{2.10}$$

As already reviewed in the introduction, the idea behind the construction of the reduced action [2, 3] is to express the corresponding equations in terms of currents only, solve the conformal conformal gauge constraints algebraically introducing a new set of field variables directly related to the currents, then choose a κ-symmetry gauge and finally reconstruct the action corresponding to the resulting field equations in terms of current variables. This construction implies the classical equivalence of the original and "reduced" sets of equations; in particular, the reduced theory is also integrable [2].

The Virasoro constraints can be solved by fixing a special G-gauge and residual conformal diffeomorphism gauge such that[8]

$$P_+ = \mu\, T, \qquad P_- = \mu\, g^{-1} T g, \qquad \mu = \mathrm{const} . \tag{2.11}$$

[8]In general, we may introduce two different parameters μ_+ and μ_- in P_+ and P_-; the resulting expression for the reduced action will then be obtained by replacing $\mu \to \sqrt{\mu_+ \mu_-}$.

Here μ is an arbitrary scale parameter (the scale corresponding to fixing the residual conformal diffeomorphisms, similar to p^+ in light-cone gauge) and T is a special constant matrix chosen in [2] to be[9]

$$T = \frac{i}{2}\begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, \qquad T^2 = -\frac{1}{4}I, \qquad \text{Str } T^2 = 0\,. \tag{2.12}$$

Here Σ is defined in (2.2) and we also introduced a new bosonic field variable g which belongs to $G = \text{Sp}(2,2) \times \text{Sp}(4)$, i.e. to the subgroup whose Lie algebra is $\widehat{\mathfrak{f}}_0$.

Having chosen T, we may define a subgroup H in G that commutes with T, $[T, h] = 0$, $h \in H$: in the present case we get $H = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$.[10] Using the gauge freedom and the equations of motion one can choose $g \in G$ and A_+, A_- taking values in the algebra $\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$ of H and defined by

$$A_+ \equiv g\mathcal{A}_+ g^{-1} + \partial_+ g\, g^{-1}\,, \qquad A_- \equiv (\mathcal{A}_-)_{\mathfrak{h}} \tag{2.13}$$

as the new independent bosonic variables [2, 32].

Next, one can impose a partial κ-symmetry gauge

$$Q_{1-} = 0\,, \qquad Q_{2+} = 0\,, \tag{2.14}$$

and then define the new independent fermionic variables

$$\Psi_1 = Q_{1+}\,, \qquad \Psi_2 = gQ_{2-}g^{-1}\,. \tag{2.15}$$

Similarly to Q_{1+} and Q_{2-}, the new variables Ψ_1 and Ψ_2 belong to $\widehat{\mathfrak{f}}_1$ and $\widehat{\mathfrak{f}}_3$, respectively. Indeed, the adjoint action of $g \in G$ separately maps the subspaces $\widehat{\mathfrak{f}}_1$ and $\widehat{\mathfrak{f}}_3$ into themselves since the algebra of G is $\widehat{\mathfrak{f}}_0$ and according to the Z_4 decomposition $[\widehat{\mathfrak{f}}_i, \widehat{\mathfrak{f}}_j] = \widehat{\mathfrak{f}}_{i+j \ mod\ 4}$. i.e. $[\widehat{\mathfrak{f}}_0, \widehat{\mathfrak{f}}_3] = \widehat{\mathfrak{f}}_3$. Note also that Ψ_1 and Ψ_2 are completely independent being related to different components of the fermionic current.

The residual κ-symmetry can be fixed by further restricting $\Psi_{1,2}$ by demanding that they anticommute with T, $\{\Psi_{1,2}, T\} = 0$. Namely, we may introduce the projector from $\Psi_{1,2}$ to $\Psi^{\parallel}_{1,2}$

$$\Psi^{\parallel} \equiv \Pi\Psi = \frac{1}{2}(\Psi + 4T\Psi T)\,, \qquad \Pi^2 = \Pi\,, \tag{2.16}$$

$$\Psi^{\parallel}T = -T\Psi^{\parallel}, \qquad [T, [T, \Psi^{\parallel}]] = -\Psi^{\parallel}, \quad \Psi^{\parallel} = [T, \widehat{\Psi}], \quad \widehat{\Psi} = -2T\Psi. \tag{2.17}$$

Note that since according to (2.2) $[\Sigma, K] = 0$ the projector Π commutes with the "reality condition" projectors $\mathcal{P}_\pm$ in (2.5), so that it can be imposed in addition to the constraints (2.6) or (2.7).

The Z_2 decomposition implied by Π can be represented explicitly as follows:

$$\Psi = \begin{pmatrix} 0 & X \\ X^\dagger\Sigma & 0 \end{pmatrix}, \qquad X = X^{\parallel} + X^{\perp}, \qquad X^{\parallel} = -\Sigma X^{\parallel}\Sigma, \qquad X^{\perp} = \Sigma X^{\perp}\Sigma\,. \tag{2.18}$$

[9]The choice of normalization of T is of course arbitrary and can be changed by rescaling μ.

[10]Note that there is a natural arbitrariness in the choice of g in eq. (2.11) since P_- is invariant under $g \to hg$ if $h \in H$; that implies an additional H gauge invariance of the resulting equations of motion for g.

Writing X in terms of 2×2 blocks and using (2.2) we get

$$X \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad X^{\parallel} = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \qquad X^{\perp} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} . \tag{2.19}$$

We may then define the new fermionic variables as [2]

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}}\Psi_1^{\parallel}, \qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}}\Psi_2^{\parallel}, \tag{2.20}$$

so that $\Psi_{_R}$ and $\Psi_{_L}$ are expressed in terms of "off-diagonal" matrices $X_{_R}$ and $X_{_L}$ as $X^{\parallel}$ in (2.19). The "reality" constraints (2.7) on $\Psi_{_R} \in \widehat{\mathfrak{f}}_1$ and $\Psi_{_L} \in \widehat{\mathfrak{f}}_3$ in (2.7) then imply that the corresponding 2×2 blocks are expressed in terms of *real* Grassmann 2×2 matrices ξ and η ($J^2 = -I$, see (2.2))

$$\beta_{_{R,L}} = \xi_{_{R,L}} \pm iJ\xi_{_{R,L}}J \ , \qquad \gamma_{_{R,L}} = \eta_{_{R,L}} \mp iJ\eta_R J \ . \tag{2.21}$$

Explicitly, in terms of 2×2 blocks

$$\Psi_{_R} = \begin{pmatrix} 0 & 0 & 0 & \xi_{_R} + iJ\xi_{_R}J \\ 0 & 0 & \eta_R - iJ\eta_{_R}J & 0 \\ 0 & -\eta^t_{_R} - iJ\eta^t_{_R}J & 0 & 0 \\ \xi^t_{_R} - iJ\xi^t_{_R}J & 0 & 0 & 0 \end{pmatrix}, \tag{2.22}$$

$$\Psi_{_L} = \begin{pmatrix} 0 & 0 & 0 & \xi_{_L} - iJ\xi_{_L}J \\ 0 & 0 & \eta_{_L} + iJ\eta_{_L}J & 0 \\ 0 & -\eta^t_{_L} + iJ\eta^t_{_L}J & 0 & 0 \\ \xi^t_{_L} + iJ\xi^t_{_L}J & 0 & 0 & 0 \end{pmatrix}. \tag{2.23}$$

Thus each of $\Psi_{_R}$ and $\Psi_{_L}$ are parametrized by $2 \times 4 = 8$ independent real Grassmann variables. Note that the change $R \to L$ is equivalent to $i \to -i$, i.e.

$$\Psi_{_R}(\xi_R, \eta_R) = \Psi^*_{_L}(\xi_L \to \xi_R, \eta_L \to \eta_R) \ . \tag{2.24}$$

2.2 Lagrangian of the reduced theory

The reduced theory Lagrangian that reproduces the classical equations of the reduced theory (obtained from first-order equations corresponding to the GS Lagrangian (2.9)) is given by the left-right symmetrically gauged WZW model for

$$\frac{G}{H} = \frac{\mathrm{Sp}(2,2)}{\mathrm{SU}(2) \times \mathrm{SU}(2)} \times \frac{\mathrm{Sp}(4)}{\mathrm{SU}(2) \times \mathrm{SU}(2)}$$

supplemented by the following integrable bosonic potential and the fermionic terms [2]:

$$\begin{aligned} L_{\rm tot} = L_B + L_F = {} & L_{\rm gWZW}(g, A) + \mu^2\, \mathrm{Str}(g^{-1}TgT) \\ & + \mathrm{Str}\left(\Psi_{_L}TD_+\Psi_{_L} + \Psi_{_R}TD_-\Psi_{_R} + \ \mu\, g^{-1}\Psi_{_L}g\Psi_{_R}\right) . \end{aligned} \tag{2.25}$$

Here all fields are represented by 8×8 supermatrices (so that Str in bosonic terms means the difference of traces of the $su(2,2)$ and $su(4)$ parts. The covariant derivative is $D_\pm\Psi =$

$\partial_\pm \Psi + [A_\pm, \Psi]$, $A_\pm \in \mathfrak{h}$. Given that $[T, h] = 0$, $h \in H$, the Lagrangian $L_{\rm tot}$ is invariant under H gauge transformations

$$g' = h^{-1} g h, \qquad A'_\pm = h^{-1} A_\pm h + h^{-1} \partial_\pm h, \qquad \Psi'_{_{L,R}} = h^{-1} \Psi_{_{L,R}} h\ . \tag{2.26}$$

The μ-dependent terms in (2.25) are essentially the original GS Lagrangian after the substitution of (2.11), (2.14), (2.15) and (2.20); one may conjecture that $L_{\rm gWZW}(g, A)$ plus free fermionic terms should originate from the change of variables (from fields to currents) in the original GS string path integral [2, 4].

Similarly to the original closed string GS action, the reduced theory action is defined on a 2d cylinder (i.e. the fields are 2π periodic in σ) and should also have the string tension in front of it. In discussing UV (short distance) behavior of the theory the compactness of the σ direction is not relevant; likewise the masses of fields are also unimportant. In that discussion we shall therefore formally replace the cylinder with coordinates (τ, σ) by a plane and consider the mass terms as part of the interaction potential. In that case the parameter μ (which, as we shall see will not be renormalized) can be set to 1 by rescaling the worldsheet coordinates; we will prefer however not to do that explicitly.

The dimension of the bosonic target space in (2.25) is the same as the dimension of the G/H coset, i.e. 4+4=8. The fermionic fields having "standard" two-dimensional fermionic kinetic terms are represented by the 8×8 matrices subject to the two Z_2 grading conditions discussed above, so that they are describing eight left-moving and eight right-moving Grassmann degrees of freedom. Remarkably, the reduced action is only quadratic in fermions, in contrast to original GS action which is at least quartic in fermions in a generic real κ-symmetry gauge.

Another way of writing the fermionic terms, which takes into account the constraint $T\Psi_{_{L,R}} = -\Psi_{_{L,R}} T$, follows from introducing an explicit projector in the fermion kinetic term, as was done in [2]: $\Psi T D \Psi \to \Psi T \Pi D \Psi$. The resulting action is[11]

$$L_F = \frac{1}{2} {\rm Str} \left(\Psi_{_L} [T, D_+ \Psi_{_L}] + \Psi_{_R} [T, D_- \Psi_{_R}] + \ 2\mu\, g^{-1} \Pi \Psi_{_L} g \Pi \Psi_{_R} \right) . \tag{2.27}$$

The second "reality" constraint (2.7) implied by the Z_4 split may also be implemented by insertion of the corresponding projectors.

One may also write the action in terms of the independent real Grassmann variables entering the explicit solution (2.22), (2.23) of the constraints. Using (2.22), (2.23) fermionic kinetic term in L_F then takes the standard simple form (upon integration by parts)[12]

$$L_{F0} = {\rm Str}(\Psi_{_L} T \partial_+ \Psi_{_L} + \Psi_{_R} T \partial_- \Psi_{_R}) = -2i\ {\rm tr}(\xi^t_{_L} \partial_+ \xi_{_L} + \eta^t_{_L} \partial_+ \eta_{_L} + \xi^t_{_R} \partial_- \xi_{_R} + \eta^t_{_R} \partial_- \eta_{_R}) \tag{2.28}$$

[11] The projectors in the interaction term may be omitted as they will be implemented in perturbation theory through the fermionic propagator factors. Another equivalent way of writing the action is solve the constraint $\{T, \Psi\} = 0$ as $\Psi = [T, \widehat{\Psi}]$.

[12] Note that (up to a total derivative) ${\rm Str}(\Psi T d\Psi) = -\frac{i}{2} {\rm tr}[X^\dagger (dX - \Sigma dX \Sigma)]$, where we used eq. (2.5) and the fact that the fermionic matrices anticommute under the ordinary trace.

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The gauge connection in $D_\pm$ which belongs to $\mathfrak{h} = su(2)\oplus su(2)\oplus su(2)\oplus su(2)$ can be easily included. If $A = \mathrm{diag}(A_1, A_2, A_3, A_4)$, $A_i \in su(2)$ then we get terms like $\mathrm{tr}[\beta^\dagger(A_1\beta - \beta A_4) - \gamma^\dagger(A_2\gamma - \gamma A_3)]$. Then the action can be rewritten in terms of independent 2×2 matrices.[13]

The "Yukawa" interaction term in (2.25) can be written in more explicit form by using that

$$g = \begin{pmatrix} g^{(1)} & 0 \\ 0 & g^{(2)} \end{pmatrix}, \qquad g^{(1)} \in \mathrm{Sp}(2,2), \quad g^{(2)} \in \mathrm{Sp}(4) \tag{2.29}$$

$$\mathrm{Str}(g^{-1}\Psi_{_L} g \Psi_{_R}) = \mathrm{tr}(g^{(1)-1} X_{_L} g^{(2)} X_{_R}^\dagger \Sigma - g^{(2)-1} X_{_L}^\dagger \Sigma g^{(1)} X_{_R}) , \tag{2.30}$$

where

$$X_{_R} = \begin{pmatrix} 0 & \xi_{_R} + iJ\xi_{_R} J \\ \eta_{_R} - iJ\eta_{_R} J & 0 \end{pmatrix}, \qquad X_{_L} = \begin{pmatrix} 0 & \xi_{_L} - iJ\xi_{_R} J \\ \eta_{_L} + iJ\eta_{_L} J & 0 \end{pmatrix}. \tag{2.31}$$

This fermionic interaction term is the only one that that mixes the bosonic fields $g^{(1)} \in \mathrm{Sp}(2,2)$ and $g^{(2)} \in \mathrm{Sp}(4)$ of the reduced models (based on gWZW models for $\frac{\mathrm{Sp}(2,2)}{\mathrm{SU}(2)\times\mathrm{SU}(2)}$ and $\frac{\mathrm{Sp}(4)}{\mathrm{SU}(2)\times\mathrm{SU}(2)}$) for the AdS_5 and S^5 parts of the original GS coset model.[14] The fermions carry representations of both $\mathrm{Sp}(2,2)$ and $\mathrm{Sp}(4)$ and thus intertwine the two bosonic sub-theories.[15]

It is this interaction that is responsible for making the reduced model UV finite, i.e. conformally invariant modulo the built-in scale parameter μ (which is the remnant of gauge-fixing the conformal diffeomorphisms at the classical level).

At the level of the equations of motion the H gauge field $A_\pm$ can be gauged away; the result is the following fermionic generalization of the non-abelian Toda equations [2] (see also [50])

$$\partial_-(g^{-1}\partial_+ g) + \mu^2[g^{-1}Tg, T] + \mu[g^{-1}\Psi_{_L} g, \Psi_{_R}] = 0\,, \tag{2.32}$$

$$\partial_-\Psi_{_R} - 2\mu T(g^{-1}\Psi_{_L} g)^\parallel = 0\,, \qquad \partial_+\Psi_{_L} - 2\mu T(g\Psi_{_R} g^{-1})^\parallel = 0\,, \tag{2.33}$$

$$(g^{-1}\partial_+ g - 2T\Psi_{_R}\Psi_{_R})_{\mathfrak{h}} = 0\,, \qquad (g\partial_- g^{-1} - 2T\Psi_{_L}\Psi_{_L})_{\mathfrak{h}} = 0\,, \tag{2.34}$$

where the last line follows from the equations for $A_\pm$ and we used that $\Psi_{_{L,R}}$ anticommute with T (see (2.16), (2.20)) as well as that $T^2 = -\frac{1}{4}I$.

One may also eliminate the gauge fields from the fermionic terms in (2.25) as usual in 2 dimensions — by writing $A_+ = u\partial_\pm u^{-1}$, $A_+ = \bar{u}\partial_\pm \bar{u}^{-1}$ and performing a local rotation

[13] Expanding near the trivial solution $A = 0$, $g = 1$ the fermionic action then takes the form equivalent to the quadratic fermionic action in the near - pp-wave or BMN limit in eqs. (5.6), (5.7) in [10].

[14] A similar term in the original GS action reflects the presence of the RR 5-form coupling.

[15] This feature resembles more a WZW models based on a supergroup rather than a supersymmetric extension of WZW model. At the same time, the fermions here have first-order kinetic term, so we obtain a kind of hybrid model. In the special case of $AdS_2 \times S^2$ the resulting reduced model does have 2d supersymmetry and is equivalent to the $\mathcal{N} = 2$ supersymmetric extension of the sine-Gordon model. In this case $G = \mathrm{SO}(1,1)\times\mathrm{SO}(2)$ so the fermions are in the singlet representation. A less trivial case of the reduced model for $AdS_3 \times S^3$ was worked out explicitly in [4]; there the existence of the 2d supersymmetry in the resulting model is not obvious and remains an open question.

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of the fermions.[16] The bosonic gWZW part of the Lagrangian written in terms of $h_\pm$ becomes $L_{\rm WZW}(u^{-1}g\bar{u}) - L_{\rm WZW}(u^{-1}\bar{u})$ and the potential term can also be written in terms of $\widetilde{g} = u^{-1}g\bar{u}$ since T commutes with $u, \bar{u}$.

Alternatively, one may fix an H gauge on g and integrate the fields $A_\pm$ out [2] leading to a bosonic sigma model with 4+4 dimensional target space coupled to 8 fermions (with quadratic and quartic fermionic terms).[17]

Since the fermions are transforming in different representation than bosons, the reduced Lagrangian (2.25) is not of a familiar supersymmetric gWZW theory (deformed by a bosonic potential and Yukawa-type terms) and thus more difficult to analyze. It is nevertheless a simple well-defined theory intimately connected to the $AdS_5 \times S^5$ GS superstring. It is therefore of interest to study its quantum properties. Finiteness of $AdS_5 \times S^5$ superstring (checked directly to the two-loop order [14]) suggests, assuming the relation via the reduction should hold beyond the classical level, that this theory should also be UV finite. In contrast to the GS superstring, here it should be much easier to verify the finiteness since the reduced theory is power counting renormalizable.

Indeed, the reduced theory is obviously UV finite for $\mu = 0$ (since gWZW model coupled to fermions is). Also, the structure of the μ-dependent interaction terms in (2.25) is constrained by symmetries, and it seems possible that bosonic and fermionic contributions to renormalization of the potential terms may cancel each other (as they do in the reduced model for $AdS_2 \times S^2$ superstring which is the $\mathcal{N} = 2$ supersymmetric sine-Gordon theory). Our aim below will be to present evidence that this model is indeed UV finite.

3 Bosonic part of the reduced theory and UV divergences

To get an idea about the structure of possible UV divergences in reduced theory (2.25) let us first consider its bosonic part. We shall first review the form of the sigma model that appears as a result of choosing a specific parametrization of the basic field $g \in G$ and integrating out the H gauge field A_a. That assumes that the H-gauge is fixed by choosing a particular form of the group element g.

In the case of the string on $R_t \times S^n$ or sigma model on the sphere $F/G = S^n$ the reduced theory is based on the gWZW model for $G/H = {\rm SO}(n)/{\rm SO}(n-1)$. It is constructed by choosing a parametrization of g in terms of the coordinates of the G/H coset and integrating out the H gauge field A_a. We end up with an integrable theory represented by an (n-1)-dimensional sigma model with a potential (see [2])

$$L = G_{mk}(x)\ \partial_+ x^m \partial_- x^k - U(x)\ . \tag{3.1}$$

[16] As in the supersymmetric WZW model, the corresponding Jacobian may lead to a shift of the coefficient of the bosonic term.

[17] A disadvantage of this gauge is that the resulting action does not allow a straightforward expansion near the $g = 1$ point. For this purpose it seems necessary to choose a "intermediate" gauge, where both $A_\pm$ and g are partially fixed.

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Here x^m represent the $n-1$ $(= \dim G - \dim H)$ independent components of g left after fixing the H gauge.[18] The potential term (or "tachyon coupling" in string sigma model language) in (3.1) originates directly from the μ^2 term in the action. It is a relevant (in the case of a compact group F such as for the sphere) or irrelevant (in the case of a non-compact group F such as for AdS_n) perturbation of the gWZW model and thus also of the "reduced" geometry, i.e. it should satisfy

$$\frac{1}{\sqrt{G}e^{-2\Phi}}\partial_m(\sqrt{G}e^{-2\Phi}G^{mk}\partial_k)U - M^2U = 0\,, \tag{3.2}$$

where Φ is the dilaton resulting from integrating out A_a. An explicit parametrization of g in the case of $G = \mathrm{SO}(n)$ in terms of Euler angles is found by choosing

$$g = g_{n-1}(\theta_{n-1})\dots g_2(\theta_2)g_1(2\varphi)g_2(\theta_2)\dots g_{n-1}(\theta_{n-1})\,, \tag{3.3}$$

where $g_m(\theta) = e^{\theta R_m}$ and $R_m \equiv R_{m,m+1}$ are generators of $\mathrm{SO}(n+1)$. Thus $\varphi \equiv \frac{1}{2}\theta_1$, and θ_p $(p = 2, \dots, n-1)$ are $n-1$ coordinates on the resulting coset space Σ^{n-1}, with φ playing a distinguished role. Then the potential U has a universal form for any dimension n: it is simply proportional to $\cos 2\varphi$ as in the sine-Gordon model $(n = 2)$ [2]. The metric and the dilaton resulting from integrating out the H gauge field A_a satisfy

$$ds^2 = G_{mk}dx^m dx^k = d\varphi^2 + \mathrm{g}_{pq}(\varphi,\theta)d\theta^p d\theta^q\,, \qquad \sqrt{G}\; e^{-2\Phi} = (\sin 2\varphi)^{n-2}\,, \tag{3.4}$$

so that the equation (3.2) is indeed solved by

$$U = -\frac{\mu^2}{2}\cos 2\varphi\,, \qquad M^2 = -4(n-1)\,, \tag{3.5}$$

i.e.

$$L = \partial_+\varphi\partial_-\varphi + \mathrm{g}_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi\;. \tag{3.6}$$

The explicit form of the Σ^{n-1} metric (3.4) with $n = 2, 3, 4$ as found directly from the (2.25) with (3.3) is the following. For the reduced models for S^2 and S^3, i.e. for $G/H = \mathrm{SO}(2)$ and $G/H = \mathrm{SO}(3)/\mathrm{SO}(2)$ we have

$$ds^2_{n=2} = d\varphi^2\,, \qquad ds^2_{n=3} = d\varphi^2 + \cot^2\varphi\; d\theta^2\;. \tag{3.7}$$

For $G/H = \mathrm{SO}(4)/\mathrm{SO}(3)$ [34]

$$ds^2_{n=4} = d\varphi^2 + \cot^2\varphi\;(d\theta_1 + Vd\theta_2)^2 + \tan^2\varphi\;\frac{d\theta_2^2}{\sin^2\theta_1}\,, \qquad V = \cot\theta_1\tan\theta_2\,, \tag{3.8}$$

[18] In contrast to the metric of the usual geometric (or "right") coset $\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1}$ the metric G_{mk} in (3.1) found from the symmetrically gauged $G/H = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ gWZW model will generically have singularities and no non-abelian isometries. The corresponding space may be denoted as Σ^{n-1}. While the gauge $A_a = 0$ preserves the explicit $\mathrm{SO}(n-1)$ invariance of the equations of motion, fixing the gauge on g and integrating out A_a breaks all non-abelian symmetries (the corresponding symmetries are then "hidden", cf. [53]). Instead of $R_{mk} = \; a\, G_{mk}$ for a standard sphere the metric G_{mk} satisfies $R_{mk} + 2\nabla_m\nabla_k\Phi = 0$ where Φ is the corresponding dilaton resulting from integrating out A_a.

or after a change of variables $x = \cos\theta_1 \ \cos\theta_2, \quad y = \sin\theta_2$

$$ds^2_{n=4} = d\varphi^2 + \frac{\cot^2\varphi \, dx^2 + \tan^2\varphi \, dy^2}{1 - x^2 - y^2} \,. \tag{3.9}$$

From $G/H = \mathrm{SO}(5)/\mathrm{SO}(4)$ gWZW we get [33]

$$ds^2_{n=5} = d\varphi^2 + \cot^2\varphi \, (d\theta_1 + V d\theta_2 + W d\theta_3)^2 + \tan^2\varphi \left(\frac{d\theta_2^2}{\cos^2\theta_1} + \frac{d\theta_3^2}{\sin^2\theta_1} \right), \tag{3.10}$$

$$V = \frac{\tan\theta_1 \sin 2\theta_2}{\cos 2\theta_2 + \cos 2\theta_3}, \qquad W = \frac{\cot\theta_1 \sin 2\theta_3}{\cos 2\theta_2 + \cos 2\theta_3} \,. \tag{3.11}$$

Together with the $\cos 2\varphi$ potential (3.5) the latter metric thus defines the reduced model for the string on $R_t \times S^5$.

One can similarly find the reduced Lagrangians for $F/G = AdS_n = \mathrm{SO}(2, n-1)/\mathrm{SO}(1, n-1)$ coset sigma models which are related to the above ones by an analytic continuation. A "mnemonic rule" to get the AdS_n counterparts of S^n reduced Lagrangians is to change $\varphi \to i\phi$ and to reverse the overall sign of the Lagrangian. In general, that will give the $G/H = \mathrm{SO}(1, n-1)/\mathrm{SO}(n-1)$ counterpart of (3.6) of the form

$$L = \partial_+\phi\partial_-\phi + \widetilde{\mathrm{g}}_{pq}(\phi, \vartheta)\partial_+\vartheta^p\partial_-\vartheta^q - \frac{\mu^2}{2}\cosh 2\phi \,, \tag{3.12}$$

where $\widetilde{\mathrm{g}}_{pq}(\phi) = -\mathrm{g}_{pq}(i\phi)$ (i.e. $\cot^2\varphi \to \coth^2\phi$ in (3.7), etc.).

The reduced model for bosonic strings on $AdS_n \times S^n$ can then be obtained by formally combining the reduced models for strings on $AdS_n \times S^1$ and on $R \times S^n$ [2]. For example, in the case of a string in $AdS_2 \times S^2$ we find the sum of the sine-Gordon and sinh-Gordon Lagrangians

$$L = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi) \,, \tag{3.13}$$

while for a string in $AdS_3 \times S^3$ we get (see [2, 4])

$$L = \partial_+\varphi\partial_-\varphi + \cot^2\varphi \, \partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \coth^2\phi \, \partial_+\vartheta\partial_-\vartheta + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi) \,. \tag{3.14}$$

Similar bosonic actions are found for a string in $AdS_4 \times S^4$ and in $AdS_5 \times S^5$ using (3.8) and (3.10).

Next, let us discuss the quantum properties of the above bosonic sigma models. Since these are deformations of conformal gWZW models, we should not expect infinite renormalization of the resulting sigma model metrics,[19] but the potential terms may get renormalized. While the $\cos 2\varphi$ potential is a relevant perturbation of the coset CFT in the compact S^n case, the $\cosh 2\phi$ is an irrelevant perturbation of the corresponding coset CFT in the AdS_n case (i.e. the sign of the mass term M^2 in (3.5) is opposite). Thus the coefficients of the two terms in the potential in (3.14) (and in similar higher-dimensional models) "run"

[19] On dimensional grounds, the deformation terms cannot contribute to the renormalization of the two-derivative terms.

in the opposite directions. As a result, the bosonic reduced theory like (3.13) or (3.14) is not renormalizable already at the leading one-loop order: one would need to introduce two different bare coefficients in front of the $\cos 2\varphi$ and the $\cosh 2\phi$ terms in the potential to cancel the divergences.

A simple way to see that different renormalization is to note that the one-loop correction given by $\log\det\Delta$ terms is not sensitive to a change of sign of the classical action which should be done while going from S^n to AdS_n reduced model via $\varphi \to i\phi$. Thus if in the S^n model we get a divergence $c_1 \cos 2\varphi \ \ln\Lambda$, then in the AdS_n model it should be given simply by the same with $\varphi \to i\phi$, i.e. by $c_1 \cosh 2\phi \ \ln\Lambda$. Hence the total divergence will be $c_1(\cos 2\varphi + \cosh 2\phi)\log\Lambda$. It will thus have a different structure than the classical potential in (3.13), (3.14), and so cannot be absorbed into renormalization of the single parameter μ.

More generally, the supertrace symbol in $\mathrm{Str}(g^{-1}TgT)$ in (2.25) means that the potential terms for the AdS_5 and S^5 parts of the reduced theory are taken with the opposite signs (i.e. as $\cos 2\varphi - \cosh 2\phi$ in the Euler angle parametrization (3.3)). Since the anomalous dimensions[20] of the corresponding two terms are opposite (which is related to the opposite signs of curvature of AdS_5 and S^5), the logarithmically divergent term coming from the bosonic part of (2.25) is actually the sum, not the difference, i.e. defined in terms of g in the product of the two groups it contains tr instead of Str

$$L_{1-\mathrm{loop}} = a_1 \ \mathrm{tr}(g^{-1}TgT) \ \ln\Lambda \ . \tag{3.15}$$

One expects that in the full reduced theory (2.25) corresponding to the $AdS_5 \times S^5$ superstring the fermionic terms will make the whole theory UV finite, i.e. (3.15) will be canceled by the fermionic contributions, i.e. the potential can be considered as an exactly marginal perturbation (with the value of its coefficient μ being finite and arbitrary).

This is indeed what happens in the $AdS_2 \times S^2$ case where the reduced theory is equivalent to the (2,2) supersymmetric sine-Gordon theory [2]. For this to happen in the general theory (2.25) the contribution to the divergences coming from the fermionic Yukawa interaction term should also be proportional to (3.15), i.e. to the sum of the bosonic potentials instead of their difference entering the classical action.

It is possible to argue that indeed the fermionic part is invariant under the analytic continuation $\varphi \to i\phi$, so that its one-loop contribution to the renormalization of the bosonic potential should also be even, i.e. proportional to the sum of the potential terms as in (3.15). For example, the explicit form of the fermionic terms in the $AdS_3 \times S^3$ case given in [4] is invariant under $\varphi \to i\phi$, $\phi \to -i\varphi$. In the next section we shall give a general argument of why that should happen and check explicitly that the resulting divergent coefficient indeed cancels against the bosonic one.

Let us continue with several general remarks about the structure of 2-loop renormalization of the potential (or "tachyon coupling") term in a generic bosonic sigma model

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \ \left[G_{mn}(x)\partial^\mu x^m \partial_\mu x^n + \epsilon^{\mu\nu}B_{mn}(x)\partial_\mu x^m \partial_\nu x^n - U(x)\right] . \tag{3.16}$$

[20] It is useful to recall that $\mathrm{tr}(g^{-1}TgT)$ is a primary field of the WZW theory [43].

The renormalization of U is governed by the β-function (see, e.g., [35–38])

$$\beta^U = -\gamma U - 2U\,, \tag{3.17}$$

$$\gamma = \Omega^{mn} D_m D_n + O(\alpha'^4)\,, \tag{3.18}$$

$$\Omega^{mn} = \frac{1}{2}\alpha' G^{mn} + p_1 \alpha'^2 R^{mn} + p_2 \alpha'^2 H^m_{\ kl} H^{nkl} + O(\alpha'^3)\,. \tag{3.19}$$

Here we follow the notation of [36, 39]. The 2-loop coefficients p_1, p_2 are scheme dependent (they can be changed by redefining G_{mn}). In dimensional regularization with minimal subtraction [35, 36] $p_1 = 0$ while p_2, in principle, still depends on how one treats $\epsilon^{\mu\nu}$ in dimensional regularization (cf. [37, 39–41]). In a scheme where $\epsilon^{\mu\nu}$ is considered as being 2-dimensional one [40] (which also corresponds to the $f_1 = -1$ scheme in [5]) one finds [37, 39] $p_2 = -\frac{1}{8}$. In this case the dilaton and tachyon 2-loop β-functions take the form[21]

$$\beta^\phi = -\gamma\phi + \frac{1}{6}\Big[D - \frac{1}{4}\alpha' H_{mkl}H^{mkl} + O(\alpha'^3)\Big], \tag{3.20}$$

$$\beta^U = -\gamma U - 2U\,, \qquad \gamma = \frac{1}{2}\alpha'\Big[G^{mn} - \frac{1}{4}\alpha' H^m_{\ kl} H^{nkl} + O(\alpha'^3)\Big] D_m D_n\,. \tag{3.21}$$

In the case of a WZW model (i.e. when the group space is a target space and H_{mkl} is the parallelizing torsion) these expressions are then in agreement with the WZW central charge ($C = 6\beta^\phi$, $\phi = \text{const}$) and the anomalous dimension of the field $\text{tr}g(\sigma)$ as found in [43] (see also [40, 44]):

$$C = \frac{kd}{k + \frac{1}{2}c_G} = d\Big(1 - \frac{c_G}{2k} + \cdots\Big)\,, \qquad \gamma U = \frac{c_r}{k + \frac{1}{2}c_G} U = \frac{c_r}{k}\Big(1 - \frac{c_G}{2k} + \cdots\Big) U\,, \tag{3.22}$$

where $\alpha' = \frac{1}{k}$, $R_{mn} = \frac{1}{4} H_{mkl} H_n^{\ kl} = \frac{R}{d} G_{mn}$, $c_G = \frac{2R}{d}$, $G^{mn}G_{mn} = d$ and c_r and c_G are the values of the Casimir operator in, respectively, the fundamental and adjoint representations. More explicitly, if we consider the renormalization of a potential term in a WZW model

$$L = L_{\rm WZW}(g) - U(g)\,, \tag{3.23}$$

as we shall do in the next section, then, as follows from the above general results, the *2-loop* renormalization of U will originate only from the vertices in the WZ term in the action (and will be, in general, scheme-dependent).

Such a 2-loop shift in the anomalous dimension is absent in 2d supersymmetric WZW models due to an additional contribution of the fermions that are chirally coupled to g. That can be seen by first integrating the fermions out which leads to the shift of the overall coefficient k of the WZW term[22] $k \to k' = k - \frac{1}{2}c_G$ and thus eliminates all higher than 1-loop contributions to the anomalous dimension of U: the corresponding dimension in (3.22) is then $\frac{c_r}{k' + \frac{1}{2}c_G} = \frac{c_r}{k}$.

[21]The corresponding operator γ enters also the dilaton β-function considered in [39]. See also the discussion around eq. (5.10) in the second reference in [47].

[22]In WZW model written in a manifestly supersymmetric form the fermions are Majorana spinors coupled to g as $\text{tr}(\bar\psi\gamma_5\gamma^\mu[\partial_\mu g g^{-1}, \psi])$, and their rotation $\psi_L \to g^{-1}\psi_L g$, $\psi_R \to g\psi_L g^{-1}$ that decouples them from g produces a non-trivial jacobian that shifts the coefficient of the WZW term [48].

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The case of the reduced theory which we shall consider below is different from the case 2d supersymmetric WZW theory with a bosonic potential in that here there is an additional fermionic interaction term that contributes to the renormalization of the bosonic potential and completely cancels out also the 1-loop anomalous dimension.

An apparent consequence of the above general expression for β^U (3.21) is that in the sigma models like (3.14) obtained by integrating out the gauge field A where there is no WZ-type B_{mn} coupling ($H_{mnk} = 0$) there will be no non-trivial renormalization of the potential at the 2-loop order. There is a caveat that since this sigma model is obtained from a conformal gWZW model its classical metric will be conformal only in a special scheme [38]; in a standard (minimal subtraction) scheme the metric will be deformed by $\alpha' = \frac{1}{k}$ corrections starting from the 2-loop order [45–47]. As a result, expressed in terms of the "tree-level" metric, the anomalous dimension will receive an effective 2-loop contribution coming from the 1-loop term after one uses there the 1-loop corrected metric. This subtlety would be absent in a 2d supersymmetric gWZW model where, as recalled above, the fermions produce a compensating shift of the level k and thus the expressions for the central charge, anomalous dimension and the effective sigma model metric obtained by integrating out the A gauge field remain essentially the 1-loop ones (see [47] and refs. therein).

Though there is no apparent 2d supersymmetry in our reduced Lagrangian (2.25) one may suspect that the effect of fermions there may be similar to the one in the 2d supersymmetric gWZW case. If we assume that the fundamental quantum variables are actually the GS fermionic currents Q_1 and Q_2 in (2.9) then (2.15) which defines $\Psi_{_L}$ and $\Psi_{_R}$ is similar to a rotation that decouples fermions from bosons and produces the level shift $k \to k' = k - \frac{1}{2}c_{_G}$ in the 2d supersymmetric WZW model.[23] The above remark does not, however, directly apply to our case since the fermionic kinetic term in (2.25) contains the matrix T which does not in general commute with g so after the rotation of $\Psi_{_L}$ we will be left with a non-trivial $g^{-1}Tg$ coupling in its kinetic term.

As was already stressed above, compared to WZW theory coupled to fermions, we have in addition a fermionic counterpart of the potential term in (2.25) that may also contribute to the renormalization of the bosonic potential. This "Yukawa" interaction term originated from the fermionic WZ term in the original GS action (2.9) and thus its contribution (beyond the 1-loop level) may be sensitive to a choice of regularization, just like the treatment of the bosonic WZ term is.

These issues are related to the fundamental question: how we actually define the quantum version of the reduced theory, i.e. which is the choice of the basic quantum variables, path integral measure and regularization? This question is especially non-trivial here in view of the absence of a manifest symmetry relating the bosonic and fermionic variables. It is natural to assume that these choices should be made so that to ensure that

[23] Indeed, the standard relation [49] for a fermionic determinant implies $\det(\partial_+ + \mathrm{Adj}_{g^{-1}\partial_+ g})\det(\partial_- + \mathrm{Adj}_{\tilde{g}^{-1}\partial_- \tilde{g}}) = \exp[c_{_G} I_{\rm WZW}(g\tilde{g}^{-1})]\ \det\partial_+\ \det\partial_-$. Here we assumed that fermions are in adjoint representation; otherwise $c_{_G}$ should be replaced by the corresponding quadratic Casimir of the representation, $T_aT_a = c_r I$. This expression can be factorized into separate chiral determinant contributions using Polyakov-Wiegmann identity, and then $I_{\rm WZW}(g)$ (or $I_{\rm WZW}(\tilde{g}^{-1})$) can be interpreted as the effective action for a Dirac fermion with purely right (left) coupling to the corresponding current.

the resulting theory is UV finite, just like the original GS theory should be.

Below we shall assume that the fundamental fermionic variables are $\Psi_{_L}$ and $\Psi_{_R}$ having canonical kinetic terms and will show that all 1-loop divergent contributions to the potential terms cancel, while the 2-loop contributions which are, in general, scheme-dependent, also vanish in a natural regularization scheme.

4 UV finiteness of the reduced theory

In this section we shall study the divergences of the reduced model (2.25) for strings in $AdS_5\times S^5$ without first integrating out the H gauge field. This allows us to utilize explicitly the conformal invariance of the gWZW model so that the only possible renormalization that needs to be analyzed is that of the potential terms.

4.1 Change of variables in the reduced action

To study the quantum properties of reduced model it is useful to reorganize its action and decouple the H gauge field as was already mentioned below eq.(2.34), i.e. following the same pattern as in the bosonic gauged WZW models. Namely, we can always choose the two-dimensional gauge fields to be of the form

$$A_+^{(i)} = u^{(i)}\partial_+ u^{(i)-1}\,, \qquad A_-^{(i)} = \bar u^{(i)}\partial_+ \bar u^{(i)-1}\,, \tag{4.1}$$

where $i=1,2$ labels the two copies of SO(4) algebra in the algebra of H isomorphic to $\mathrm{SO}(4)\times\mathrm{SO}(4)$. Then, the coupling between g and the gauge field may be eliminated by redefining $g=\mathrm{diag}(g^{(1)},g^{(2)})\in \mathrm{Sp}(2,2)\times\mathrm{Sp}(4)$ as follows

$$\widetilde g^{(i)} = u^{(i)-1} g^{(i)} \bar u^{(i)}\ . \tag{4.2}$$

This redefinition may be written more compactly as $\widetilde g = u^{-1} g \bar u$ by introducing the "super-matrices"[24]

$$\widetilde g = \begin{pmatrix} \widetilde g^{(1)} & 0 \\ 0 & \widetilde g^{(2)} \end{pmatrix}\,, \qquad u = \begin{pmatrix} u^{(1)} & 0 \\ 0 & u^{(2)} \end{pmatrix}\,, \qquad \bar u = \begin{pmatrix} \bar u^{(1)} & 0 \\ 0 & \bar u^{(2)} \end{pmatrix}\ . \tag{4.3}$$

We can also redefine the fermionic fields in (2.25) as[25]

$$\widetilde\Psi_{_L} = u^{-1}\Psi_{_L} u\,, \qquad \widetilde\Psi_{_R} = \bar u^{-1}\Psi_{_R}\bar u\ . \tag{4.4}$$

Then the reduced Lagrangian (2.25) becomes

$$\begin{aligned} L = {}& L^{(G)}_{\rm WZW}(\widetilde g) - k' L^{(H)}_{\rm WZW}(u^{-1}\bar u) + \mu^2 \mathrm{Str}\big(\widetilde g^{-1} T \widetilde g T\big) \\ & + \mathrm{Str}\big(\widetilde\Psi_{_L} T\partial_+ \widetilde\Psi_{_L} + \widetilde\Psi_{_R} T\partial_- \widetilde\Psi_{_R}\big) + \mu \mathrm{Str}\big(\widetilde g^{-1}\widetilde\Psi_{_L}\widetilde g \widetilde\Psi_{_R}\big)\ . \end{aligned} \tag{4.5}$$

We used that $u\in H$ commutes with T. Here the factor k' in the second term indicates the shift of the overall coefficient (or the level k, that we formally set to 1) coming from the

[24] The supertrace of such matrices is defined as a difference of traces of diagonal blocks.

[25] Note that since $u,\bar u$ are from H and thus commute with T the rotated fermionic fields also satisfy the constraints in (2.16), (2.20), i.e. they anticommute with T.

Jacobians of the above change of variables from $A_\pm$ to $u, \bar{u}$ and from the rotations of the fermions (4.4) as in the usual 2d supersymmetric gWZW case [30]. Here the shift is $k' = k + (1 - \frac{1}{2})c_{so(4)}$ where $c_{so(4)}$ is the quadratic Casimir of $H^{(1)} = \mathrm{SO}(4)$. The shift by $c_{so(4)}$ is coming from the bosonic Jacobian and by $-\frac{1}{2}c_{so(4)}$ from the chiral fermionic Jacobians regularized in a vector-like fashion so that their contributions combine into $L^{(H)}_{\rm WZW}(u^{-1}\bar{u})$.

This redefinition is very useful for the purpose of studying the UV properties of the theory: we can ignore the decoupled WZW term for the subgroup H (i.e. the term multiplied by k' in (4.7)) since it is conformally invariant on its own. The fermions in (4.7) have free kinetic terms. By formally assuming that T transforms under $G = \mathrm{Sp}(2,2) \times \mathrm{Sp}(4)$ in an appropriate way[26] we may then treat the remaining terms in the action as being invariant under G.

Let us note that in general one can not, of course, completely decouple $L_{\rm WZW}(u^{-1}\bar{u})$ term: the gauge-invariant observables in the original theory may depend on u and $\bar{u}$. Indeed, the action (4.7) — even written in an apparently factorized form — still exhibits the following gauge invariance

$$\widetilde{g} \mapsto h\widetilde{g}h^{-1}\,, \qquad \widetilde{\Psi}_{L,R} \mapsto h\widetilde{\Psi}_{L,R}h^{-1}\,, \qquad u \mapsto huh^{-1}\,, \qquad \bar{u} \mapsto h\bar{u}h^{-1}\,, \qquad (4.6)$$

where $h = \mathrm{diag}(h^{(1)}, h^{(2)}) \in \mathrm{SO}(4) \times \mathrm{SO}(4)$. The observables of this theory must be invariant under these transformations. Clearly, traces of products of powers of $\widetilde{g}$ and T are invariant. However, partial derivatives of $\widetilde{g}$ must be promoted to covariant derivatives of $\widetilde{g}$. Thus, u and $\bar{u}$ must necessarily enter the observables.

4.2 Structure of divergences in quantum effective action

We are interested in understanding the UV finiteness properties of the theory (2.25) or, equivalently, of (4.5). To simplify the notation in what follows we shall omit tildes on g and Ψ in (4.5), i.e. study the UV properties of the following theory

$$L = L^{(G)}_{\rm WZW}(g) + \mu^2 \mathrm{Str}\big(g^{-1}TgT\big) + \mathrm{Str}\big(\Psi_{_L} T\partial_+ \Psi_{_L} + \Psi_{_R} T\partial_- \Psi_{_R}\big) + \mu \mathrm{Str}\big(g^{-1}\Psi_{_L} g \Psi_{_R}\big)\,, \quad (4.7)$$

where $g \in \mathrm{Sp}(2,2) \times \mathrm{Sp}(4)$.

This theory is power counting renormalizable but it is not clear a priori that divergences will preserve the specific structure of the potential terms. Indeed, as was discussed in the previous section, the bosonic part of (4.7) is the sum of the two decoupled theories for $g^{(1)} \in \mathrm{Sp}(2,2)$ and $g^{(2)} \in \mathrm{Sp}(4)$ with the potential terms "running" in the opposite directions. Thus renormalizability of the bosonic theory a priori would require us to add also the coupling (see (3.15)) $\widetilde{\mu}^2 \mathrm{tr}\big(g^{-1}TgT\big)$ or introduce two independent couplings for the two bosonic potentials.

Moreover, fermionic coupling constant in (4.7) need not be equal (in the absence of explicit 2d supersymmetry) to the square of the coupling in the bosonic potential, i.e. it

[26] One may define this transformation as follows. The fixed matrix T identifies an $\mathrm{SO}(4)\times\mathrm{SO}(4)$ subgroup of $\mathrm{Sp}(2,2)\times\mathrm{Sp}(4)$. Then, $\mathrm{Sp}(2,2)\times\mathrm{Sp}(4)$ transformations of T amount to choosing different (but equivalent) embeddings $\mathrm{SO}(4) \times \mathrm{SO}(4) \subset \mathrm{Sp}(2,2) \times \mathrm{Sp}(4)$. At the level of the original action, a realization of this symmetry requires transformations of the gauge field. This is not surprising, given that one gauges different $\mathrm{SO}(4) \times \mathrm{SO}(4)$ subgroups of $\mathrm{Sp}(2,2) \times \mathrm{Sp}(4)$.

may be some μ' that may "run" differently than μ.[27] Our analysis below shows that the corresponding 1-loop renormalization group equations admit a fixed point $\mu' = \mu$, $\widetilde{\mu} = 0$, i.e. with this choice all 1-loop divergences (including the ones depending on fermions) cancel. As for the 2-loop divergences, their coefficients happen, in general, to be scheme dependent and there exists a scheme where they are absent, providing strong evidence of the finiteness of the theory (4.7).

We will study the divergent part of the effective action $\Gamma[g]$ for the bosonic field g obtained by expanding the fields around some generic background g (solving the classical equations of motion)

$$g \to g\, e^{\zeta}\,, \qquad g^{-1} \to e^{-\zeta}\, g^{-1}\,, \tag{4.8}$$

and integrating out the fluctuation field ζ (taking values in the algebra of G) and the fermions. Let us discuss the expected structure of this effective action. It should be consistent with all the global symmetries which are:

1. manifest $G = \mathrm{Sp}(2,2) \times \mathrm{Sp}(4)$ symmetry *assuming* that one treats T as a field transforming in the bifundamental representation. As mentioned above, this symmetry is manifest at the level of the classical action (4.7).
2. symmetry under formal rescaling $g \mapsto ag$ which simply means that each term in the classical action contains an equal number of factors of g and of g^{-1}.[28]
3. invariance under $g \leftrightarrow g^{-1}$, $\Psi_{_L} \leftrightarrow \Psi_{_R}$ combined with the world-sheet $\sigma_+ \leftrightarrow \sigma_-$ transformation.
4. $g^{(i)} \mapsto (-1)^{a_i} g^{(i)}$, $\Psi_{_{L,R}} \mapsto (-1)^{b_{L,R}} \Psi_{_{L,R}}$, with $a_1, a_2, b_{_L}, b_{_R} = 0, 1$ and $a_1 + a_2 + b_{_L} + b_{_R} = 2$.
5. $g^{(1)} \leftrightarrow g^{(2)}$, $\Psi_{_L} \leftrightarrow \Psi_{_R}$ (interchanging the off-diagonal blocks in the fermionic matrices in (2.22), (2.23)) together with changing the sign of the Lagrangian, i.e. the sign of the overall coupling constant.

The contributions to the effective action depend on either $j_\pm = g^{-1}\partial_\pm g$ if they come from the WZW action or explicitly g if they come from the μ-dependent (or "deformation") terms in (4.7). Two-dimensional Lorentz invariance requires that all factors of the vector $j_\pm$ appear in pairs. The structure of the action (4.7) (in particular, the chiral symmetry of the WZW model) and the fact that j has dimension 1 imply that the coefficient of the j^2 term must be finite (generated by diagrams containing at least two propagators). For that reason below we will concentrate on the derivative-independent terms built out of g.

[27] It is easy to see on dimensional grounds that quartic fermionic terms (which are a priori possible to put into the bare action) are not actually induced here with UV divergent coefficients and thus their coefficients can be set to zero.

[28] Since $g = \mathrm{diag}(g^{(1)}, g^{(2)})$ is an element of $\mathrm{Sp}(2,2) \times \mathrm{Sp}(4)$ this formal rescaling takes us outside the domain of definition of g so we will understand this rescaling only in the sense of counting the numbers of g and g^{-1} factors.

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The symmetries (a) and (b) above imply that at each loop order the effective action $\Gamma[g]$ is a combination of the tr and Str of polynomials in $g^{-n}Tg^nT$. The symmetry (c) implies that in each monomial g always appears raised to the same power as its inverse. The symmetry (d) implies that the number of factors of g plus the number of factors g^{-1} in each term is even. Finally, the symmetry (e) together with the fact that $g = \text{diag}(g^{(1)}, g^{(2)})$ is block-diagonal imply that the contribution to the effective action from diagrams with an *even* number of loops is the *supertrace* of a polynomial in $g^{-n}Tg^nT$ while the contribution from diagrams with an *odd* number of loops is the *trace* of a polynomial in $g^{-n}Tg^nT$ (cf. (3.15)).

Since the only bare g factors may come from the potential terms, having more than two factors of g and g^{-1} requires having more than two vertices from the μ-dependent terms. The number of factors of μ produced this way equals the total number of factors of g plus the number of factors of g^{-1}. Then the only way to obtain the correct dimension of the effective action is to ensure that the coefficients of such terms are given by (two-dimensional) momentum integrals with negative mass dimension; such integrals are finite in the UV.

From the arguments above it follows that the only potentially divergent contributions to the bosonic part of the effective action must be proportional to μ^2 *before* the momentum integrals are evaluated. Divergences of this type may be proportional to either the bosonic potential term in (2.25), i.e. $\text{Str}[g^{-1}TgT]$ in (4.7), or to $\text{tr}[g^{-1}TgT]$. Such contributions may come from the two types of diagrams: diagrams with one vertex from the bosonic potential and diagrams with two vertices from the boson-fermion ("Yukawa") interaction term in (4.7).[29]

In the following all integrals will be defined with an implicit IR regulator which is different from the UV regulator. This is needed since we are interested only in UV divergences. In this regime, masses of particles are irrelevant. In other words, we can expand in powers of the mass parameter of the world sheet fields or in powers of μ.

A special trick that we shall use below to simplify the calculation of the UV divergences is to treat the field g (and the fluctuation field ζ) as unconstrained matrices rather than elements (of the algebra) of $Sp(2,2) \times Sp(4)$. This is possible to do by assuming that the matrix multiplication in the action contains factors of the symplectic $Sp(2,2)$ and $Sp(4)$ metrics. Such factors project out the non-$Sp(2,2) \times Sp(4)$ parts of the fields in each term of the action. Effectively, the contraction with the symplectic metric introduces the appropriate projectors in vertices and propagators.

To define the perturbation theory we will need the propagators for the bosonic fluctuation fields ζ in (4.8) and the fermionic fields that can be parametrized as ($\chi_{L,R}$ and $\lambda_{L,R}$ are 4×4 matrices expressed in terms of $\xi_{L,R}$ and $\eta_{L,R}$, see (2.12), (2.22), (2.23))

$$[T, \Psi_{L,R}] = \begin{pmatrix} 0 & \lambda_{L,R} \\ \chi_{L,R} & 0 \end{pmatrix} . \tag{4.9}$$

[29] An equivalent argument can be given of course by starting directly with the action (2.25). Depending on the number of loops one may have additional vertices arising from the expansion of the action. Due to its gauge invariance, the gauge field in the gauged WZW action can only contribute through its field strength, so on dimensional grounds it cannot contribute to the UV-divergent terms proportional to μ^2.

(a) (b)

Figure 1. One-loop diagrams contributing to the logarithmic divergences. Bosonic propagators are denoted by solid lines and fermionic ones by dashed lines. Black dots denote vertices coming from the bosonic and the bosonic-fermionic potential term in the classical action (4.7).

We will use $(a, b, \dots)$ for the $\mathrm{Sp}(2,2)$ indices and $(\bar{a}, \bar{b}, \dots)$ for the $\mathrm{Sp}(4)$ indices and introduce the corresponding symplectic metrics

$$\Omega_{ac}\Omega^{bc} = \delta_a^b \,, \qquad \Omega_{\bar{a}\bar{c}}\Omega^{\bar{b}\bar{c}} = \delta_{\bar{a}}^{\bar{b}} \,. \tag{4.10}$$

Then the bosonic propagator is

$$\langle \zeta_{ab}\zeta_{cd}\rangle = \frac{a_{_b}}{p^2}(\Omega_{ac}\Omega_{bd} + \Omega_{ad}\Omega_{bc}) \,, \qquad \langle \zeta_{\bar{a}\bar{b}}\zeta_{\bar{c}\bar{d}}\rangle = -\frac{a_{_b}}{p^2}(\Omega_{\bar{a}\bar{c}}\Omega_{\bar{b}\bar{d}} + \Omega_{\bar{a}\bar{d}}\Omega_{\bar{b}\bar{c}}) \,, \tag{4.11}$$

and the fermionic one is ($p_\pm = p_0 \pm p_1$)

$$\langle \lambda_{L\,a\bar{b}}\chi_{L\,\bar{c}d}\rangle = \frac{i\,a_{_f}}{p_+}(T_{ad}\Omega_{\bar{b}\bar{c}} - T_{\bar{b}\bar{c}}\Omega_{ad}) \,, \qquad \langle \chi_{R\,\bar{c}d}\lambda_{R\,a\bar{b}}\rangle = \frac{i\,a_{_f}}{p_-}(T_{\bar{c}\bar{b}}\Omega_{da} - T_{da}\Omega_{\bar{c}\bar{b}}) \tag{4.12}$$

Here $a_{_b}$ and $a_{_f}$ are normalization constants

$$a_{_b} = -\frac{1}{4} \,, \qquad a_{_f} = \frac{1}{2} \,, \tag{4.13}$$

which we shall sometimes keep arbitrary for generality.

4.3 1-loop order

The 1-loop contribution to the effective action $\Gamma[g]$ is given simply by the logarithm of the ratio of the determinants of the bosonic and fermionic kinetic operators in the g-background. To test its finiteness it is enough to show the cancellation of the first two terms in the μ-expansion of the logarithm of these determinants.

The leading (μ-independent power-like divergent) term in the expansion simply counts the difference between the number of bosonic and fermionic degrees of freedom and thus cancels automatically. To demonstrate the cancellation of the subleading (logarithmic) divergence requires a short calculation. The relevant Feynman diagrams are shown in figure 1.

These diagrams represent the next-to-leading order in the mass μ expansion of the trace of logarithm of the bosonic and fermionic kinetic operators. Their cancellation tests the mass sum rule for the fluctuation fields

$$\sum_i (-1)^{f_i} m_i^2 = 0 \,. \tag{4.14}$$

The vertices in figure 1 arise from the expansion of the bosonic and the fermionic terms in the action (4.7) (g here is the background field)

$$L_2^{(b)} = \frac{1}{2}\mu^2 \mathrm{Str}\left[\left(\zeta^2 T + T\zeta^2 - 2\zeta T\zeta\right) g^{-1} T g\right] \,, \tag{4.15}$$

$$L_2^{(f)} = \mu \mathrm{Str}\left[\Psi_{_R} g^{-1} \Psi_{_L} g\right] \,. \tag{4.16}$$

We shall formally assume that the fields have $\mathrm{Sp}(n-2,2)\times\mathrm{Sp}(n)$-valued indices (we will set $n=4$ at the end). Then the relevant contribution of the bosonic diagram to the effective action is

$$L^{(b)}_{1-\text{loop}} = \mu^2 a_{_b}\left(\frac{n+1}{2}+\frac{n+1}{2}-1\right) I_1 \ \mathrm{tr}[g^{-1}TgT]\,, \qquad\qquad I_1=\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2}\,, \tag{4.17}$$

where tr is the *trace* over $\mathrm{Sp}(n-2,2)\times\mathrm{Sp}(n)$ indices and in the integral I_1 we assume the presence of both UV and IR cutoffs.[30] In what follows we shall use dimensional ($d=2-2\varepsilon$) UV regularization, and the IR divergences can be subtracted as, e.g., in [41, 55] by replacing the massless propagators by $\frac{1}{p^2}\to\frac{1}{p^2}+\frac{\pi}{\varepsilon}\delta^{(2)}(p)$.

The three terms in the bracket in (4.17) came from the three terms in $L_2^{(b)}$ in (4.15). We used that (cf. (4.11))

$$\langle\zeta^2_{ad}\rangle \equiv \langle\zeta_{ab}\Omega^{bc}\zeta_{cd}\rangle = \frac{a_{_b}}{p^2}(1+n)\Omega_{ad}\,, \qquad \langle\zeta^2_{\bar a\bar d}\rangle \equiv \langle\zeta_{\bar a\bar b}\Omega^{\bar b\bar c}\zeta_{\bar c\bar d}\rangle = -\frac{a_{_b}}{p^2}(1+n)\Omega_{\bar a\bar d}\,,$$
$$\langle(\zeta T\zeta)_{ah}\rangle \equiv \langle\zeta_{ab}\Omega^{bc}T_{cd}\Omega^{de}\zeta_{eh}\rangle = -\frac{a_{_b}}{p^2}T_{ah}\,, \tag{4.18}$$

and that T with *two lower indices* (i.e. with one index lowered by Ω) is an antisymmetric matrix. The fermionic contribution is

$$L^{(f)}_{1-\text{loop}} = \frac{1}{2}\mu^2 a^2_{_f}\big(n+n\big)\ I_1\ \mathrm{tr}[g^{-1}TgT]\,, \tag{4.19}$$

where in the denominator of the integral we used that $-p_+p_-=p^2$ and the overall $\frac{1}{2}$ came form the expansion of the logarithm of the kinetic operator to the second order. To arrive at (4.19) we noted that decomposing each vertex in 4×4 blocks transforming in the representations of $\mathrm{Sp}(n-2,2)\times\mathrm{Sp}(n)$ one finds two terms for each vertex. Each term in one vertex contracts with exactly one term in the second vertex and each contraction yields one of the two terms in the bracket in (4.19).

Adding $L^{(b)}_{1-\text{loop}}$ (4.17) and $L^{(f)}_{1-\text{loop}}$ (4.19) one observes that they cancel out (since according to (4.13) $a_{_b}=-a^2_{_f}=-\frac{1}{4}$). This implies that (4.14) is indeed satisfied and thus the 1-loop effective action for a generic classical background g is finite.

Similarly, one may show also the non-renormalization of the fermionic interaction term in (4.7), implying the cancellation of the 1-loop correction to the fermionic propagator. Note that in the $AdS_2\times S^2$ case the presence of two-dimensional supersymmetry in the reduced action [2] makes this calculation redundant, but in general we do not know which symmetry (if any) relates the bosonic and the fermionic potential terms in the reduced Lagrangian (2.25). Since these two terms appeared (after gauge fixing and field

[30] Thus the 1-loop bosonic anomalous dimension of the operator $\mathrm{tr}[g^{-1}TgT]$ in $G=\mathrm{Sp}(n)$ WZW theory is proportional to n. This coefficient is different from the dimension of $\mathrm{tr}g$ which is proportional to $n+1$ (see (3.22), $c_{_r}(\mathrm{Sp}(n))=n+1$). From the general perspective of the sigma model anomalous dimension in (3.21) this difference can be attributed to the difference of eigenvalues of the Laplace operator on the group space when acting on the corresponding operators. To compute the action of the Laplacian on $\mathrm{tr}[g^{-1}TgT]$ one may follow [44] and use that $\partial_a g = gE^m_a T_a$ and $T_aT_a=c_{_r}\mathbb{1}$ as well as a relation for $T_a[T,T_a]$ similar to the one appearing in (4.18) (this additional contribution leads to the subtraction of 1 from $c_{_r}=n+1$).

redefinition) from the original GS action (2.9) where their coefficients were related by κ-symmetry this non-renormalization effectively checks the consistency of the reduction procedure at the quantum level.

To check that there is no renormalization of the fermionic potential in (4.7) we should consider the diagram containing a single bosonic loop and an interaction vertex coming from the expansion of the fermionic interaction term to second order in the bosonic fluctuations:

$$L_{\rm int}^{(f)} = \mu {\rm Str}\Big[g^{-1}\Psi_{_L} g\ \ (\zeta^2\Psi_{_R} - \zeta\Psi_{_R}\zeta + \Psi_{_R}\zeta^2)\Big]\ . \tag{4.20}$$

The bosonic propagators (4.11) and the fact that the fermions transform in the bifundamental representation of $Sp(2,2)\times Sp(4)$ imply that the expectation value of the second term in the bracket in (4.20) vanishes identically. Finally, the sign difference between the expectation values in the first line of equation (4.18) implies that the contributions of the remaining two terms cancel each other. Indeed, we get

$$\begin{aligned}&(g^{-1}\Psi_{_L} g)^{a\bar b}\Big(\langle(\zeta^2)_{\bar b\bar c}\rangle\Psi_{_R}^{\bar c d}\Omega_{da} + \Omega_{\bar b\bar c}\Psi_{_R}^{\bar c d}\langle(\zeta^2)_{da}\rangle\Big)\\ &-(g^{-1}\Psi_{_L} g)^{\bar a b}\Big(\langle(\zeta^2)_{bc}\rangle\Psi_{_R}^{c\bar d}\Omega_{\bar d\bar a} + \Omega_{bc}\Psi_{_R}^{c\bar d}\langle(\zeta^2)_{\bar d\bar a}\rangle\Big)\end{aligned} \tag{4.21}$$

where each line represents one of the two terms of the supertrace, and then the sign difference between $\langle(\zeta^2)_{\bar b\bar c}\rangle$ and $\langle(\zeta^2)_{da}\rangle$ in (4.18) implies that each parenthesis vanishes identically.

4.4 2-loop order

Let us now proceed to analyzing the 2-loop divergent contributions to the action in (4.7). We shall ignore the power divergences.[31] The $\ln^2\Lambda$ (or double-pole) divergences should cancel (according to the standard argument) due to the cancellation of the logarithmic divergences at the 1-loop order established above. The main issue will thus be the $\ln\Lambda$ (or single pole) divergences. We shall first consider corrections to bosonic potential and then discuss possible divergent contributions to the fermionic Yukawa term.

4.4.1 Contributions to bosonic potential

The relevant diagrams (that may produce potentially divergent order μ^2 contributions) contain one μ^2-vertex from the bosonic potential or two μ-vertices from the bosonic-fermionic interaction term; they are shown in figure 2.

The first diagram contains one parity-even 4-point vertex from $L_{\rm WZW}$ in (4.7) (we shall suppress the overall $\frac{k}{4\pi}$ factor)

$$L_{{\rm WZW}\,(4)} = -\frac{1}{12}\eta^{\mu\nu}{\rm Str}\left[\left[\partial_\mu\zeta,\,\zeta\right],\left[\partial_\nu\zeta,\,\zeta\right]\right] \tag{4.22}$$

and an insertion of a 2-point vertex from the bosonic potential ("mass insertion"). As in (4.15) in eq.(4.22) ζ is assumed to be a matrix in the algebra of $Sp(n-2,2)\times Sp(n)$ (we

[31] They are absent in dimensional regularization and in any case should cancel due to the balance of degrees of freedom, the mass sum rule (4.14) or under an appropriate choice of the path integral measure.

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Figure 2. Two-loop diagrams at order μ^2. Bosonic propagators are denoted by solid lines and fermionic ones by dashed lines.

will again set $n = 4$ at the end). Namely, it is a symmetric matrix when written with both lower indices (i.e. with the upper index contracted with the symplectic metric Ω). The corresponding contribution to the effective action is proportional to the tadpole integrals:

$$L^{(a)}_{2-\text{loop}} = \hbar \frac{2\mu^2}{3} a_b^3 \; n(n+2) \; [I_1(\varepsilon)]^2 \; \text{Str}[g^{-1}TgT], \tag{4.23}$$

$$I_1(\varepsilon) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} \;, \qquad\qquad d = 2 - 2\varepsilon \;. \tag{4.24}$$

$\hbar \equiv \frac{4\pi}{k}$ is the inverse of the coefficient in front of the classical WZW action. We assumed dimensional regularization.[32]

The second diagram, containing one vertex from the bosonic potential, also yields only tadpole integrals. The bosonic 4-vertex arising from the expansion of the bosonic potential is

$$L_{\text{pot}\,(4)} = \mu^2 \text{Str}\left(\left[\frac{1}{4!}(\zeta^4 T + T\zeta^4) - \frac{1}{3!}(\zeta^3 T\zeta + \zeta T\zeta^3) + \frac{1}{(2!)^2}\zeta^2 T\zeta^2 \right] g^{-1} T g \right), \tag{4.25}$$

where the multiplication of matrices is assumed with the symplectic metric. Also, the propagator in (4.11) enforces the condition that ζ belongs to the algebra of $\text{Sp}(n-2,2) \times \text{Sp}(n)$. As was already mentioned above, this implies that we may formally treat ζ as an unconstrained matrix rather than an element of the algebra of $\text{Sp}(n-2,2) \times \text{Sp}(n)$.

The contribution of each of the three terms in (4.25) to the divergent part in the case when the group is $\text{Sp}(n)$ is proportional to

$$2 \times \frac{1}{4!}(n+1)(2n+1) - 2 \times \frac{1}{3!}[-(2n+1)] + \frac{1}{(2!)^2}[(n+1)^2 - (n+1) + 1] \;. \tag{4.26}$$

This expression holds also if we replace $\text{Sp}(n)$ by $\text{Sp}(n-2,2)$. Then the resulting divergent contribution to the bosonic potential term in the effective action is

$$L^{(b)}_{2-\text{loop}} = \hbar \frac{\mu^2}{12} a_b^2 \; n(5n-2) \; [I_1(\varepsilon)]^2 \; \text{Str}[g^{-1}TgT] \;. \tag{4.27}$$

As was already mentioned above, while at odd number of loops the divergent contributions from individual diagrams are proportional to $\mu^2 \text{tr}[g^{-1}TgT]$, at even number of loops the divergent contributions are proportional to $\mu^2 \text{Str}[g^{-1}TgT]$, i.e. have the same form as the classical potential.

[32] As we are interested in isolating the UV divergence, we understand this integral as having an implicit IR cutoff separate from the dimensional regulator, e.g., one may carry out an IR subtraction at the level of the propagators as was already mentioned above.

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Next, there is a divergent contribution from a diagram (c) with two cubic vertices from the WZ term in the WZW Lagrangian (4.7) and with a μ^2 insertion from the potential. Up to a normalization factor $\hbar^{-1} = \frac{k}{4\pi}$ common to the parity-even part of the WZW Lagrangian, the cubic interaction term is

$$L_{\mathrm{WZW}\,(3)} = \frac{2}{3}\,\epsilon^{\mu\nu}\mathrm{Str}[\zeta\,\partial_\mu\zeta\,\partial_\nu\zeta] \ . \tag{4.28}$$

It then yields

$$L^{(c)}_{2-\mathrm{loop}} = 16\hbar\mu^2 a_{_b}^4\ n(n+2)\ I_2\ \mathrm{Str}[g^{-1}TgT]\,, \tag{4.29}$$

$$I_2 \equiv -\int \frac{d^dp d^dq}{(2\pi)^{2d}}\frac{(\epsilon^{\mu\nu}p_\mu q_\nu)^2}{p^2q^2[(p+q)^2]^2} \ . \tag{4.30}$$

Here we again assumed continuation $d = 2-2\varepsilon$ but we need to decide how to treat $\epsilon_{\mu\nu}$ in dimensional regularization. This is a well-known issue (see, e.g., [39–42, 57]). In general, different regularization prescriptions may lead to different results — the coefficient of the 2-loop logarithmic divergences may be scheme-dependent, with different results related by redefinitions of the coupling constants [39, 60].

Similarly to the original GS action (2.9) containing the fermionic WZ term, the reduced action (2.25) or (4.5) does not admit a straightforward d-dimensional generalization. This is analogous to (chiral) supersymmetric theories (see, e.g., [52, 57, 59]) where it is natural to use the version of *dimensional regularization by dimensional reduction* [51]. We shall discuss alternative regularization schemes in appendix A and draw an analogy with the case of 2d supersymmetric sigma models in appendix B.

Under this prescription we shall do all Lorentz (and spinor) algebra in 2 dimensions and continue to d dimensions only scalar momentum integrals. In particular, we shall use the 2-dimensional relation

$$\epsilon^{\mu\nu}\epsilon^{\mu'\nu'} = -\eta^{\mu'\mu}\eta^{\nu'\nu} + \eta^{\nu'\mu}\eta^{\mu'\nu}\,, \tag{4.31}$$

where in the Minkowski signature notation $\eta^{\mu\nu} = (-1, 1)$. Under this prescription

$$-\,(\epsilon^{\mu\nu}p_\mu q_\nu)^2 = p^2q^2 - (p\cdot q)^2\,, \tag{4.32}$$

and thus continuing to $d = 2-2\varepsilon$ dimensions we find

$$I_2 = \int \frac{d^dp d^dq}{(2\pi)^{2d}}\frac{p^2q^2-(p\cdot q)^2}{p^2q^2[(p+q)^2]^2} = \frac{1}{4}[I_1(\varepsilon)]^2 \ . \tag{4.33}$$

The contribution of the diagram (c) in (4.29) is then given by

$$L^{(c)}_{2-\mathrm{loop}} = 4\hbar\mu^2 a_{_b}^4\ n(n+2)\ [I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}TgT] \ . \tag{4.34}$$

Adding together (4.23), (4.27) and (4.29) and using that $a_{_b} = -\frac{1}{4}$ we find that the contribution of the bosonic 2-loop diagrams to the UV singular part of 2-loop effective Lagrangian is

$$L^{\mathrm{bose}}_{2-\mathrm{loop}} = \hbar\frac{\mu^2}{32}\ n^2\ [I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}TgT]\,, \tag{4.35}$$

where

$$[I_1(\varepsilon)]^2 = \left[\frac{1}{4\pi\varepsilon} + \mathcal{O}(1)\right]^2 = \frac{1}{(4\pi)^2\varepsilon^2} + \cdots \ . \tag{4.36}$$

The coefficient of the most singular term is consistent with the expected renormalization group behavior of the bosonic theory, i.e. it is related to the square of the coefficient of the 1-loop single-pole in (4.17). The coefficient of the 2-loop subleading $\frac{1}{\varepsilon}$ pole is, in general, scheme dependent; in the standard minimal subtraction scheme we then get no genuine 2-loop divergence (i.e. the 2-loop anomalous dimension coefficient vanishes).

Let us introduce the renormalization constant $Z^{(i)}$, $i = 1, 2$, for the two bosonic operators $U^{(i)}$ corresponding to two factorized parts (related to the two subgroups of $\mathrm{Sp}(n-2,2)\times\mathrm{Sp}(n)$) in $\mu^2\mathrm{Str}[g^{-1}TgT]$, i.e. $U^{(i)} = Z^{(i)}U^{(i)}_{\rm bare}$

$$Z = \mu^{-2\varepsilon}\left[1 + \hbar\frac{\gamma_1}{\varepsilon} + \hbar^2\left(\frac{\gamma_2}{2\varepsilon} + \frac{\gamma_1^2}{2\varepsilon^2}\right) + \cdots\right], \tag{4.37}$$

where we suppressed the index i and we have chosen μ to be the renormalization scale parameter. Then the corresponding anomalous dimension is

$$\gamma = \frac{dZ^{-1}}{d\ln\mu} = 2\varepsilon + \hbar\gamma_1 + \hbar^2\gamma_2 + \cdots \ . \tag{4.38}$$

From (4.17) it is easy to see that $\gamma_1^{(1,2)} = \pm\frac{1}{16\pi}n$ which, when squared, reproduces the coefficient of the $\frac{1}{\varepsilon^2}$ pole in (4.35).

Let us now consider the fermionic contributions to the 2-loop divergent part of the bosonic effective action. There are several types of μ^2 terms which arise from bose-fermi interaction term in (4.7) and they correspond to the diagrams $2(d)$ and $2(e)$. They can be represented symbolically as coming from the square of the interacting terms in the action:

$$\begin{aligned} &2\times\frac{1}{2}\langle\int d^2\sigma\ \mathrm{Str}[g^{-1}\Psi_{_L}g\Psi_{_R}]\int d^2\sigma\ \mathrm{Str}\Big[g^{-1}\Psi_{_L}g\ \left(\zeta^2\Psi_{_R} - \zeta\Psi_{_R}\zeta + \Psi_{_R}\zeta^2\right)\Big]\rangle \\ &+\frac{1}{2}\langle\left(\int d^2\sigma\ \mathrm{Str}[g^{-1}\Psi_{_L}g\,(\zeta\Psi_{_R} - \Psi_{_R}\zeta)]\right)^2\rangle \ . \end{aligned} \tag{4.39}$$

The terms in the first line, diagram $2(d)$, lead to vanishing contributions to the logarithmic divergences either because of impossibility of proper Wick contractions (as in the second term in the brackets) or because of $\mathrm{Str}\mathbf{1} = 0$ (as in the case of the first and the third term).[33] The remaining non-trivial contribution comes from the term in the second line of (4.39), i.e. diagram $2(e)$

$$L^{(e)}_{2-\rm loop} = \hbar\mu^2 a_{_b} a_{_f}^2\ 2\times\frac{1}{2}\left[n(n+1) - n\right]\ I_3\ \mathrm{Str}[g^{-1}TgT]\,, \tag{4.40}$$

$$I_3 = \int\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{p_+q_-}{p^2q^2(p+q)^2}\,, \tag{4.41}$$

[33] This is essentially the same calculation which implies the non-renormalization of the fermionic potential at 1-loop order.

where we took into account the minus signs due to the fermionic loop, due to the supertrace in eq. (4.39) and due to the factors of i in the fermionic propagators. The $\frac{1}{2}$ factor is inherited from the last line of eq. (4.39) and the overall factor of 2 is present because the relevant contribution comes from the cross term in the square.

Here again there is an ambiguity in defining the integral I_3, i.e. in extending the factor p_+q_- in the integrand (which has its origin in the chiral nature of the fermion coupling in (4.7)) to d dimensions. In the GS action, the fermionic current components were 2d vectors and they were reinterpreted as 2d Weyl spinors in the reduced theory. The fermionic interaction term in the reduced theory (2.25) originated from the WZ term in the GS action (2.9), which suggests that chiral fermions should be treated as if they were 2-dimensional fields. An analogy with the 2d supersymmetric gWZW model suggests again to use the regularization by dimensional reduction.

Explicitly, that means that we shall first use that in 2 dimensions

$$p_+q_- = (p_0 + p_1)(q_0 - q_1) = -(\eta^{\mu\nu} + \epsilon^{\mu\nu})p_\mu q_\nu \ . \tag{4.42}$$

Equivalently, interpreting Ψ_L and Ψ_R in (4.7) as upper/lower components of left/right MW 2d spinor and rewriting the fermionic terms using the 2-component notation with the explicit 2d γ-matrix factors we observe that p_+q_- in I_3 in (4.40) arises from

$$p_+q_- = -\text{tr}[\not p \not q \frac{1}{2}(1+\gamma_3)] = -p\cdot q - \epsilon^{\mu\nu}p_\mu q_\nu \, , \tag{4.43}$$

where $\gamma_3 = \gamma_0\gamma_1$ and we assumed that all spinor algebra is done in 2 dimensions.[34]

Observing that the term with a single factor of the antisymmetric tensor $\epsilon^{\mu\nu}$ can not contribute to the integral and continuing the scalar integrand to d dimensions we end up with

$$\begin{aligned} I_3 &= -\int d^dp d^dq \frac{p\cdot q}{p^2q^2(p+q)^2} = -\frac{1}{2}\int d^dp d^dq \frac{(p+q)^2 - p^2 - q^2}{p^2q^2(p+q)^2} \\ &= \frac{1}{2}[I_1(\varepsilon)]^2 \ . \end{aligned} \tag{4.44}$$

Then finally (using (4.13))

$$L^{\text{fermi}}_{2-\text{loop}} = -\frac{1}{32}\hbar\mu^2 \ n^2 \ [I_1(\varepsilon)]^2 \ \text{Str}[g^{-1}TgT] \ . \tag{4.45}$$

Combining this with the bosonic contribution in (4.35) we conclude that the two contributions cancel each other, i.e. the bosonic part of the 2-loop effective action is UV finite,

$$L^{(\text{bos.pot.})}_{2-\text{loop}} = L^{\text{bose}}_{2-\text{loop}} + L^{\text{fermi}}_{2-\text{loop}} = \text{finite} \ . \tag{4.46}$$

As already mentioned above, this is just a reflection of the cancellation of the 1-loop logarithmic divergences as all simple $\frac{1}{\varepsilon}$ poles in both the bosonic and the fermionic contributions computed in the dimensional reduction scheme come together with a $\frac{1}{\varepsilon^2}$ pole which is controlled by the 1-loop divergences.

[34] Same result for the parity-even term is found if we extended momenta and γ-matrices to d dimensions by assuming that $\not p = \bar p^\mu \bar\gamma_\mu + \widehat p^\mu \widehat\gamma_\mu \, , \quad \{\bar\gamma^\mu, \gamma_3\} = 0 \, , \ [\widehat\gamma^\mu, \gamma_3] = 0$, where $\bar\mu$ are 2-dimensional and $\widehat\mu$ are -2ε dimensional indices, i.e. $\mu = (\bar\mu, \widehat\mu)$.

4.4.2 Contributions to fermionic potential term

The above observation, that the 2-loop correction to renormalization of the bosonic potential is scheme dependent, may seem to contradict the standard lore: in view of the cancellation of the one-loop renormalization of the potential, one could expect that the two-loop renormalization should be scheme independent being the first non-vanishing correction. However, as discussed in section 3 and below eq.(4.7), the reduced theory, when viewed as a power-counting renormalizable model, is actually a multi-coupling theory (with the level k and several μ-parameters as its couplings, with the action (4.7) corresponding to a fixed-point choice). In such a case the 2-loop anomalous dimension coefficients may still be scheme-dependent.

As was already mentioned, several a priori distinct parameters in the action were set to be equal as required by the reduction procedure starting from the GS action where they were related by symmetries. In the bosonic part of the theory these were the couplings of the two potential terms corresponding to $\mathrm{Sp}(n-2,2)$ and $\mathrm{Sp}(n)$. With fermions included, the coefficients of the bosonic and the fermionic potential terms, $\mathrm{Str}[g^{-1}TgT]$ and $\mathrm{Str}[g^{-1}\Psi_L g\Psi_R]$, were also related. It is then necessary to ensure that such relations survive quantum corrections.

As we have found above, the corrections to the bosonic potential are finite in a special dimensional reduction scheme. Finiteness of the full theory then requires that corrections to the fermionic potential be finite in that same scheme. In the apparent absence of worldsheet supersymmetry which would relate the bosonic and the fermionic potentials (and thus their renormalization, assuming one uses a supersymmetry-preserving regularization scheme) this is not a priori guaranteed.[35]

It is therefore crucial to test the finiteness of the corrections to the fermionic potential in (4.7)

$$U_f = \mu \mathrm{Str}\big(g^{-1}\Psi_{_L} g \Psi_{_R}\big) \tag{4.47}$$

in the same dimensional reduction scheme.

On dimensional grounds, to (logarithmically) renormalize U_f we need terms with a single power of μ. Since all the fermionic interactions in (4.7) are proportional to μ and the bosonic potential is proportional to μ^2, it follows that this renormalization is entirely governed by the bosonic $\mathrm{Sp}(n-2,2)\times\mathrm{Sp}(n)$ WZW model with fermions treated as background fields.

The relevant diagrams are shown in figure 3.

The computation of their divergent parts is formally similar to that of the renormalization of the bosonic potential in (4.7), assuming one treats T as a background field. There are, however, certain differences related to the different algebraic structure of T and Ψ, which prevent the bosonic results from being immediately used here. Nevertheless, the mere

[35]It is, however, important to recall again that the bosonic and the fermionic potentials are closely connected to the kinetic and WZ terms in original Green-Schwarz action where the relation between their coefficients is a consequence of the κ-symmetry. It is possible that a global remnant of the κ symmetry that may be surviving in the gauge (2.14) offers a sufficient protection to guarantee this relation to all orders in perturbation theory in the reduced model.

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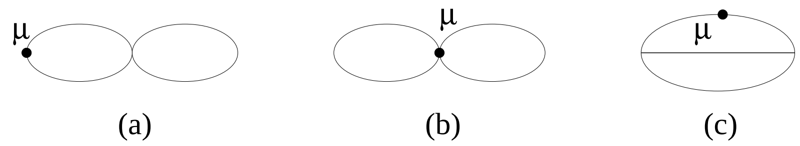

Figure 3. Two-loop diagrams contributing to renormalization of the fermionic potential. Solid lines are bosonic propagators and external fermionic legs at each μ-vertex are suppressed.

fact that the calculation is effectively governed by the undeformed $\mathrm{Sp}(n-2,2)\times\mathrm{Sp}(n)$ WZW model guarantees already that the same scheme dependence which entered the bosonic calculation will enter here as well.

Upon using the fact that $\Psi_{_{L,R}}$ are off-diagonal (transforming in bi-fundamental representation of G, see (2.22), (2.23), (4.9)) and that g is diagonal (cf. (4.3)), it is easy to see that the fermionic potential may be written as

$$U_f = \mu\Big(\mathrm{tr}[g^{(1)\,-1}\lambda_{_L} g^{(2)}\chi_{_R}] - \mathrm{tr}[g^{(2)\,-1}\chi_{_L} g^{(1)}\chi_{_R}]\Big) \ , \tag{4.48}$$

where $g^{(1)} \in \mathrm{Sp}(n-2,2)$ and $g^{(2)} \in \mathrm{Sp}(n)$. Since the $\mathrm{Sp}(n-2,2)$ and $\mathrm{Sp}(n)$ WZW models are coupled only through the μ-dependent fermionic terms, it follows that, for the purpose of the renormalization of U_f, we may treat $g^{(1)}$ and $g^{(2)}$ separately. Thus, in a diagram of topology 3(a) the fields propagating in the two loops must be of the same type since the quartic vertex coming from the WZW action involves fields of only one type (there are two distinct diagrams in this class). In a diagram of topology 3(b) the fields propagating in the two loops may be either of the same type or of different types (there are three distinct diagrams in this class). In a diagram of topology 3(c) the fields propagating in the two loops must be of the same type (there are two distinct diagrams in this class).

The diagrams of these three topologies contribute as follows to the 2-loop effective Lagrangian:

$$\begin{aligned}
L^{(a)}_{2-\mathrm{loop}} &= \hbar\mu\Big[\frac{1}{3}a_{_b}^3(n+1)(n+2) + (-1)\frac{1}{3}(-a_{_b})^3(n+1)(n+2)\Big][I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}\Psi_{_L} g\Psi_{_R}]\\
L^{(b)}_{2-\mathrm{loop}} &= -\hbar\mu\frac{a_{_b}^2}{12}(n+1)(n+2)[I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}\Psi_{_L} g\Psi_{_R}]\\
L^{(c)}_{2-\mathrm{loop}} &= \hbar\mu\left[8a_{_b}^4(n+1)(n+2) + 8(-a_{_b})^4(n+1)(n+2)\right] I_2(\varepsilon)\ \mathrm{Str}[g^{-1}\Psi_{_L} g\Psi_{_R}]\\
&= \hbar\mu\left[2a_{_b}^4(n+1)(n+2) + 2(-a_{_b})^4(n+1)(n+2)\right]\ [I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}\Psi_{_L} g\Psi_{_R}]\,,
\end{aligned} \tag{4.49}$$

where the integrals $I_1(\varepsilon)$ and $I_2(\varepsilon)$ were defined in eqs. (4.24) and (4.30), respectively, and in the last line we used eq. (4.33) relating I_2 and $(I_1)^2$.

It is interesting to note that each one of the above three contributions is proportional to $(n+1)(n+2)$. This factor may be understood on the group theory grounds as being the product of the two quadratic Casimirs, in the fundamental and the adjoint representations of $\mathrm{Sp}(n-2,2)$ or $\mathrm{Sp}(n)$. This n dependence is different from that of the corrections to the bosonic potential because, on the one hand, in the bosonic calculation one uses that (see (2.12)) $T^2 = -\frac{1}{4}\mathbb{1}$ while here the analogous quantities are $\Psi^2_{_L}$ or $\Psi_{_L}\Psi_{_R}$ do not have similar properties, and, on the other hand, some Wick contractions here are forbidden as the fields belong to different algebras.

Adding together the above three singular contributions in (4.49) we conclude, in complete analogy with the bosonic potential case, that they cancel out, i.e. the result is UV finite,

$$L_{2-\text{loop}}^{(\text{fermi.pot.})} = \text{finite} \,. \tag{4.50}$$

5 Concluding remarks

The reduced model (2.25) [2, 3] we discussed above is naturally associated, through the Pohlmeyer reduction, to the $AdS_5 \times S^5$ GS superstring action (2.9) and has certain unique features.

Its construction is based on first-order or phase space formulation of superstring dynamics in terms of supercoset currents, with the Virasoro constraints explicitly solved in terms of a new set of variables related locally to currents and thus non-locally to the original GS $AdS_5 \times S^5$ supercoset coordinates. Although various steps in the reduction do not appear to manifestly preserve 2d Lorentz invariance, the resulting reduced Lagrangian describes the dynamics of the physical number of degrees of freedom in a manifestly Lorentz invariant way. Being formulated in terms of left-invariant currents, the reduced theory is apparently "blind" to the original global PSU$(2,2|4)$ symmetry; however, being integrable (the Lax pairs of the original and the reduced theory are gauge-equivalent), it still has an infinite number of commuting charges associated to hidden symmetries, some of which are implicitly related to the global symmetries of the original GS theory.

In general, the Pohlmeyer reduction procedure, utilizing the classical conformal symmetry of a 2d sigma model, is expected to lead to an equivalent theory only at the classical level; for example, the original and reduced theory are obviously not equivalent at the quantum level if the original sigma model has a running coupling. In the present case of $AdS_5 \times S^5$ superstring sigma model, which is a conformal 2d theory at the quantum level, the relation between the original and the reduced theory has a perfect chance to hold also at the quantum level. The necessary condition for that is that the reduced theory is also UV finite.

As we have demonstrated in the present paper, the reduced theory associated to the $AdS_5 \times S^5$ superstring model is indeed free of 2d UV divergences in a certain renormalization scheme. An advantage of the reduced theory compared to the GS model is that here the main "kinetic" part of the action is based on a gauged WZW theory and thus is guaranteed to be finite; then what remains to check is only the absence of divergent contributions to the derivative-independent "potential" part of the action. We explicitly checked that at the 1-loop and 2-loop order but most likely this should be true to all orders and should be due to a hidden 2d (super)symmetry of the reduced theory.[36] The cancellation of divergences is due to a very special balance between the bosonic potential term and the fermionic

[36] If the reduced theory does not actually have a standard global 2d supersymmetry, this finiteness property suggests that there may be other similar models without 2d supersymmetry that are still UV finite. It would be interesting to classify them.

interaction term in (2.25). These two terms originated from the "kinetic" P^2 and the fermionic WZ Q^2 terms in the GS action (2.9) where they were related by κ-symmetry. This suggests that some (global) remnant of the κ-symmetry still present after fixing the κ-symmetry gauge in the reduced action may be responsible for its UV finiteness.

This opens up a possibility of solving the quantum $AdS_5 \times S^5$ superstring theory in terms of the the quantum reduced theory. The precise prescription for translating observables between the two theories remains to be understood. The most optimistic scenario is to find a path integral version of the reduction procedure based on changing the variables from coordinates to currents and solving the conformal gauge constraints as delta-function conditions $T_{++} = 0,\ T_{--} = 0$ in the path integral.

To test the equivalence of the two partition functions one may consider comparing their values for equivalent classical solutions. We leave the study of this problem for the future. Among other open problems let us mention the construction of the (2d Lorentz-invariant) S-matrix for scattering of the massive elementary excitations in the reduced theory and the determination of its relation to the BMN (magnon) S-matrix in the $AdS_5 \times S^5$ string theory in a light-cone gauge.

Let us finish with few comments on the role of the μ parameter in the reduced theory. The original GS string theory in conformal gauge has a residual part of the 2d diffeomorphism group — conformal reparametrizations — being preserved by quantum corrections. In the process of constructing the reduced theory we fix this residual symmetry by a gauge choice (cf. (2.11)) that introduces the constant parameter μ. This parameter is a fiducial scale, similar to the constant p^+ in the standard light-cone gauge.[37] Thus μ is similar to a gauge-fixing parameter and physical observables should not depend on it. For example, the expression for the energy of a particular string state expressed in terms of conserved charges of the reduced theory (or, e.g., Casimirs of the original GS global symmetry group) should not depend on μ, i.e. μ can be eliminated by re-expressing it in terms of the charges. At the same time, the S-matrix of elementary excitations with mass μ (which, by itself, is not a physical observable) will depend on μ.[38]

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[37] Indeed, the condition $P_+ = \mu T$ in (2.11) is reminiscent of the relation $\partial_+ x^+ \sim p^+$ in the light-cone gauge. Compared to standard 2d conformal theories where the infinite-dimensional conformal group is interpreted as a global symmetry imposed through conditions on physical states, in the context of string theory this is part of the 2d diffeomorphism gauge symmetry and one is allowed to fix it by a gauge choice.

[38] One may draw an analogy with quantization of strings in plane wave background. In conformal gauge one has a sigma model with target space metric like $ds^2 = dx^+dx^- + ax_ix_idx^+dx^+ + dx_idx_i$ and certain global symmetry group. One may, in principle, develop a covariant quantization and find the spectrum of states which will be classified by charges of that symmetry. We may instead fix the light-cone gauge $x^+ = p^+\tau$ and obtain a model containing free bosons (and fermions, as in the pp-wave model [23, 24, 56] associated to $AdS_5 \times S^5$ background) with mass $\mu = p^+$. Then the spectrum will depend on that μ, but we may re-interpret that dependence as that on one of the global charges which has a fixed value (proportional to μ) in that light-cone gauge.

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A Comments on regularization scheme ambiguity

Regularization scheme dependence of the 2-loop corrections to the bosonic and fermionic potentials implies that while apparently different results may be obtained under different choices of regularization (and, in particular, of treatment of fermions and Levi-Civita tensors), all of them are related by suitable redefinitions of the coupling constants of the theory. The most natural regularization scheme should be consistent with the symmetries of the theory, and we believe the dimensional reduction regularization used in the main text is such a scheme, though that seems non-trivial to demonstrate explicitly.[39] For completeness, in this appendix we discuss the 2-loop results in some alternative regularization schemes.

A version of dimensional regularization prescription (which *does not*, however, preserve the d-dimensional Lorentz invariance) is to continue momenta to $d = 2 - 2\varepsilon$ from the very beginning while still treating the Levi-Civita tensor $\epsilon^{\mu\nu}$ as if it is defined only in 2-dimensions [40] (i.e. $\epsilon^{\mu\nu} \to \bar{\epsilon}^{\mu\nu} \equiv \epsilon^{\bar{\mu}\bar{\nu}}$, $\bar{\mu}, \bar{\nu} = 1, 2$). Then instead of (4.32) we get

$$\begin{aligned} -(\bar{\epsilon}^{\mu\nu} p_\mu q_\nu)^2 &= \bar{p}^2\bar{q}^2 - (\bar{p}\cdot\bar{q})^2 = [(p^2 - \widehat{p}^2)(q^2 - \widehat{q}^2) - (p\cdot q - \widehat{p}\cdot\widehat{q})^2] \\ &= [p^2q^2 - (p\cdot q)^2] - [p^2\widehat{q}^2 + q^2\widehat{p}^2 - 2p\cdot q\,\widehat{p}\cdot\widehat{q}] + [\widehat{p}^2\widehat{q}^2 - (\widehat{p}\cdot\widehat{q})^2]. \end{aligned} \tag{A.1}$$

Here $\bar{p}$ and $\widehat{p}$ are the 2-dimensional and -2ε-dimensional components of the momentum p in $d = 2 - 2\varepsilon$ dimensions, $p^\mu = (p^{\bar{\mu}}, p^{\widehat{\mu}})$. The contribution of the first square bracket in the last line to the integral in (4.30) is then the same as in (4.33), while the second bracket leads to

$$\begin{aligned} &-\int \frac{d^dp d^dq}{(2\pi)^{2d}} \frac{[p^2\widehat{q}^2 + q^2\widehat{p}^2 - 2p\cdot q\,\widehat{p}\cdot\widehat{q}]}{p^2q^2[(p+q)^2]^2} = -\int \frac{d^dp d^dq}{(2\pi)^{2d}} \frac{\widehat{p}\cdot\widehat{q}}{p^2q^2(p+q)^2} \\ &= -\frac{\widehat{\eta}_{\mu\nu}\eta^{\mu\nu}}{d} \int \frac{d^dp d^dq}{(2\pi)^{2d}} \frac{p\cdot q}{p^2q^2(p+q)^2} = -\frac{\epsilon}{d}[I_1(\varepsilon)]^2 \,, \end{aligned} \tag{A.2}$$

where we used that for $d = 2 - 2\varepsilon$ one has $\widehat{\eta}_{\mu\nu}\eta^{\mu\nu} = -2\varepsilon$. The integral of the remaining square bracket in the last line of (A.1) may be written as

$$(\widehat{\eta}_{\mu\nu}\widehat{\eta}_{\rho\sigma} - \widehat{\eta}_{\mu\rho}\widehat{\eta}_{\nu\sigma}) \int \frac{d^dp d^dq}{(2\pi)^{2d}} \frac{p^\mu p^\nu q^\rho q^\sigma}{p^2q^2[(p+q)^2]^2} \,, \tag{A.3}$$

and produces a finite $O(\varepsilon^2)[I_1(\varepsilon)]^2$ contribution. As a result, the expression for I_2 in (4.30) in this regularization scheme is given by the sum of (4.33), (A.2) and (A.3), i.e.

$$I_2 = \left[\frac{1}{4} - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2)\right][I_1(\varepsilon)]^2 \,. \tag{A.4}$$

[39] An intuitive reason is that the reduced model is related to the $AdS_5 \times S^5$ GS superstring where the κ-symmetry should be preserved. The 2-loop finiteness of the $AdS_5 \times S^5$ superstring demonstrated in [14] in this scheme is a strong indication in this direction.

The contribution of the diagram (c) in (4.29) is then

$$L^{(c)}_{2-\text{loop}} = 4\hbar\mu^2 a_b^4\ n(n+2)\ \Big[1 - 2\mathrm{b}\varepsilon + \mathcal{O}(\varepsilon^2)\Big] [I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}TgT]\,, \tag{A.5}$$

where $\mathrm{b} = 0$ in the dimensional reduction regularization used in section 4.4.1 with I_2 given by (4.33) and $\mathrm{b} = 1$ in the second regularization prescription where I_2 is given by (A.4). The total bosonic contribution is then

$$L^{\text{bose}}_{2-\text{loop}} = \frac{\hbar\mu^2}{32}\ \left[n^2 - \mathrm{b}n(n+2)\varepsilon + \mathcal{O}(\varepsilon^2)\right]\ [I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}TgT]\,, \tag{A.6}$$

where $\mathrm{b} = 0$ corresponds to (4.35).

Thus, unlike what happened in the regularization by dimensional reduction, the bosonic contribution to the 2-loop anomalous dimension does not vanish in this (d-dimensional Lorenz-violating) scheme. The resulting value for the 2-loop anomalous dimension is, however, in agreement with the standard expression for the two-loop anomalous dimension in a sigma model with a WZ coupling (see discussion below eq.(3.17)) and, in particular, with the expression for the anomalous dimension of the primary field tr$g$ in WZW theory [43] in (3.22).[40]

Similarly to the treatment of I_2 there are several options of how to define the integral I_3 (4.41), i.e. of how to extend it to d dimensions. Instead of using the dimensional reduction scheme we may choose to extend momenta to d dimensions from the start but treat the indices of the integrand factor p_+q_- in (4.41) as 2-dimensional ones. Then instead of (4.42) we have ($\bar\mu, \bar\nu = 1, 2$)

$$p_+q_- = (p_0 + p_1)(q_0 - q_1) = -\eta^{\bar\mu\bar\nu} p_{\bar\mu} q_{\bar\nu} - \epsilon^{\bar\mu\bar\nu} p_{\bar\mu} q_{\bar\nu}\,, \tag{A.7}$$

and computing the integral in (4.41) gives, instead of (4.44),[41]

$$I_3 = -\int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{(\bar p \cdot \bar q)}{p^2 q^2 (p+q)^2} = \frac{1}{2}\left(1 + \frac{2\varepsilon}{d}\right) [I_1(\varepsilon)]^2\ . \tag{A.8}$$

Then

$$L^{\text{fermi}}_{2-\text{loop}} = -\frac{\hbar\mu^2}{32}\ n^2(1 + \mathrm{f}\varepsilon)\ [I_1(\varepsilon)]^2\ \mathrm{Str}[g^{-1}TgT]\,, \tag{A.9}$$

where $\mathrm{f} = 0$ corresponds to the dimensional reduction prescription used in (4.45) and $\mathrm{f} = 1$ corresponds to the above prescription leading to (A.8).

---

[40] In the bosonic theory with the group $\mathrm{Sp}(n-2,2)\times\mathrm{Sp}(n)$ we have the kinetic and potential terms for each factor decoupled, so that for, e.g., $G = \mathrm{Sp}(n)$ we get for the two anomalous dimensions, cf. (4.37), (4.38) ($c_G = c_{\mathrm{Sp}(n)} = n + 2$) $\gamma(\mathrm{Sp}(n-2,2)) = \frac{c_1}{k}(-1 + \frac{c_G}{2k} + \cdots)$, $\gamma(\mathrm{Sp}(n)) = \frac{c_1}{k}(1 + \frac{c_G}{2k} + \cdots)$, where $c_1 = c_r = n + 1$ (=Casimir of the fundamental representation of $\mathrm{Sp}(n)$) in the case of the tr$g$ operator and $c_1 = n$ in the present case of the $\mathrm{tr}(g^{-1}TgT)$ operator (cf. [54, 62]; for comparison, in the case of $\mathrm{tr}(g^{-1}T^agT^b)$ where T^a are generators of G one has $c_1 = c_G$ [43]). Going from one group factor to another is thus equivalent to $k \to -k$ (notice that we had Str in the WZW kinetic term in (2.25) and (4.7)).

[41] One more option is to use the straightforward dimensional regularization where $\langle p_\mu q_\nu\rangle = \frac{1}{d}\eta_{\mu\nu}\langle p\cdot q\rangle$ and thus $\langle p_+q_-\rangle = -\frac{2}{d}\langle p\cdot q\rangle$. In this case $I_3 = \frac{1}{d}[I_1(\varepsilon)]^2$ leading to $\frac{1}{2}(1+2\varepsilon)\frac{1}{(4\pi)^2\varepsilon^2}$ divergent term.

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Combining this with the bosonic contribution in (A.6) we conclude that the leading $\frac{1}{\varepsilon^2}$ singularity cancels out between the bosonic and the fermionic terms, just as the corresponding $\frac{1}{\varepsilon}$ singularity did at one loop, and we are left with

$$L^{(\text{bos.pot.})}_{2-\text{loop}} = L^{\text{bose}}_{2-\text{loop}} + L^{\text{fermi}}_{2-\text{loop}} = -\frac{\hbar\mu^2}{32}\Big[[\text{b}n(n+2)+\text{f}n^2]\varepsilon + \mathcal{O}(\varepsilon^2)\Big]\,[I_1(\varepsilon)]^2\,\text{Str}[g^{-1}TgT]$$
$$= -\frac{\hbar\mu^2}{32(4\pi)^2\varepsilon}[\text{b}n(n+2)+\text{f}n^2]\;\text{Str}[g^{-1}TgT] + \text{finite}. \quad \text{(A.10)}$$

This remaining divergent term is clearly regularization-scheme dependent and may be set to zero by an appropriate *finite* redefinition of the couplings (in particular, the level of the WZW model).

B Analogy with 2d supersymmetric sigma models with potentials

It is important to note that the dimensional reduction scheme in which the reduced model is 2-loop finite is also the scheme that would preserve 2d supersymmetry, if it were present at the classical level.

It is useful to draw analogy with a general analysis of 2-loop renormalization of (p,q) supersymmetric models deformed by potentials [58] carried out in [59]. A special case of the model considered in [59] is the (1,1) supersymmetric theory generalizing a supersymmetric WZW model to the presence of a potential term [58] (cf. (3.16))

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\Big[(G_{mn}(x)+B_{mn}(x))\partial_+x^m\partial_-x^n + iG_{mn}(x)\psi^m_L D^{(+)}_+\psi^n_L + iG_{mn}(x)\psi^m_R D^{(-)}_-\psi^n_R$$
$$+\, 2\mu D^{(-)}_m W_n(x)\psi^m_L\psi^n_R - \mu^2 G^{mn}(x)W_m(x)W_n(x)\Big]\,. \quad \text{(B.1)}$$

Here G_{mn} and B_{mn} correspond to a group space G, x^m are coordinates on G, $D^{(\pm)}$ are covariant derivatives with respect to the two "flat" connections $\Gamma^m_{nk}(G) \pm \frac{1}{2}H^m{}_{nk}(B)$,[42] and a vector W_m defines the bosonic potential.

In general [58], $W_m = U_m - V_m$, where $D_{(m}V_{n)} = 0$ (i.e. V_m is a Killing vector), $\partial_{[m}U_{n]} = \frac{1}{2}H_{mnk}V^k$, $U_mV^m = 0$. The condition of 1-loop (and, in fact, 2-loop) finiteness of such model is [59] $D_mW^m = \text{const}$.

In the simplest case $W_m = \partial_m\mathcal{W}$ where $\mathcal{W}$ is real (1,1) superpotential. In that case the action (B.1) can be written in the superfield form:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma d^2\theta\Big[(G_{mn}(X)+B_{mn}(X))\widehat{D}_+X^m\widehat{D}_-X^n - \mathcal{W}(X)\Big]\,, \quad \text{(B.2)}$$

where $X^m = x^m + \theta_+\psi^m_L + \theta_-\psi^m_R + \theta_+\theta_-F^m$ and $\widehat{D}$ are spinor derivatives.[43]

[42] As is well known, the kinetic terms of the fermions can be decoupled from bosons by defining the tangent space components like $\psi^a = E^a_m(x)\psi^m$ and "rotating" ψ^a.

[43] The (1,1) supersymmetric WZW action can also be written explicitly in terms of a superfield generalizing the group element g field [61]. Explicitly, we may replace $g = e^x$ by $\widehat{g} = e^X$, $X(\sigma,\theta) = x + \theta_+\psi_L + \theta_-\psi_R + \theta_+\theta_-F$. Then to supersymmetrize the potential $\text{tr}(g^{-1}TgT)$ we need to find the corresponding real superpotential $\mathcal{W}$. This step is straightforward for coset sigma models of the type (3.14)

In the 2d theory (B.1) the bosonic and the fermionic potential terms renormalize simultaneously, i.e. the β-functions of the corresponding couplings are related by a supersymmetry Ward identity. As was shown in [59], the 2-loop correction to this β-function vanishes in the dimensional reduction scheme similar to the one used here in section 4.4.1. Thus in the (1,1) supersymmetric theory (B.1) and the reduced theory (4.7) both treated in the dimensional reduction scheme there are no genuine 2-loop simple-pole UV divergences, all of them being accompanied by a double-pole counterpart related to single-pole 1-loop divergences as dictated by the renormalizability of the theory.

The model (4.7) based on $G = G_1 \times G_2$ bosonic WZW model with a potential coupled to fermions in bi-fundamental representations does not admit the standard version of (1,1) 2d supersymmetry: the standard supersymmetric extension of its bosonic part would be of the form (B.2), i.e. having the same number of the fermionic degrees of freedom but transforming in the adjoint representation of G. The corresponding G_1 and G_2 supersymmetric models would be mutually non-interacting and the divergences in their potential terms will not cancel, precluding finiteness.

The non-trivial property of the reduced model observed here is the cancellation of the 1-loop divergences, which makes the theory (at least) 2-loop finite. Such finiteness property is also characteristic of $(2,2)$ supersymmetric models [59]. The existence of a finite $(2,2)$ supersymmetric extension of a bosonic WZW model (with a group G which is a complex manifold) perturbed by a potential appears to be subtle and we are not aware of its discussion in the literature.[44]

References

[1] K. Pohlmeyer, *Integrable Hamiltonian systems and interactions through quadratic constraints*, *Commun. Math. Phys.* **46** (1976) 207 [SPIRES].

[2] M. Grigoriev and A.A. Tseytlin, *Pohlmeyer reduction of $AdS_5 \times S^5$ superstring σ-model*, *Nucl. Phys.* **B 800** (2008) 450 [arXiv:0711.0155] [SPIRES].

[3] A. Mikhailov and S. Schäfer-Nameki, *Sine-Gordon-like action for the superstring in $AdS_5 \times S^5$*, *JHEP* **05** (2008) 075 [arXiv:0711.0195] [SPIRES].

[4] M. Grigoriev and A.A. Tseytlin, *On reduced models for superstrings on $AdS_n \times S^n$*, *Int. J. Mod. Phys.* **A 23** (2008) 2107 [arXiv:0806.2623] [SPIRES].

whose potential depends on only two special fields φ and ϕ such that the (1,1) superpotential may be written as $\cosh\widehat{\phi} + \cos\widehat{\varphi}$ or as $\mathrm{Re}[\cos(\widehat{\varphi} + i\widehat{\phi})]$. Note that the holomorphic superpotential of the (2,2) sine-Gordon model found [2] in the special case of the model (3.13) is $\mathrm{W} = \cos(\widehat{\varphi} + i\widehat{\phi})$, but more general models like (3.14) do not admit a straightforward (2,2) extension as φ and ϕ enter separately in the two factors of the target space metric.

[44] The existence of a (2,2) superpotential deformation for the supersymmetric WZW models discussed in this appendix is, to some extent, questionable. Indeed, the relevant superpotential should be a holomorphic function on the target space. However, the target space here is factorized with one factor being compact, implying that any holomorphic function on this part of the target space is constant. More general approaches to the construction of supersymmetric extensions of sigma models with torsion encounter difficulties due to the rather trivial topology of semi-simple groups. We thank G. Papadoupoulos for useful comments on these issues.

[5] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, Nucl. Phys. **B 533** (1998) 109 [hep-th/9805028] [SPIRES].

[6] R.R. Metsaev, C.B. Thorn and A.A. Tseytlin, *Light-cone superstring in AdS space-time*, Nucl. Phys. **B 596** (2001) 151 [hep-th/0009171] [SPIRES].

[7] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in AdS_5*, Phys. Lett. **B 543** (2002) 81 [hep-th/0206103] [SPIRES].

[8] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, Phys. Rev. **D 69** (2004) 046002 [hep-th/0305116] [SPIRES].

[9] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$*, Commun. Math. Phys. **263** (2006) 659 [hep-th/0502226] [SPIRES].

[10] T. Klose, T. McLoughlin, R. Roiban and K. Zarembo, *Worldsheet scattering in $AdS_5 \times S^5$*, JHEP **03** (2007) 094 [hep-th/0611169] [SPIRES].

[11] G. Arutyunov and S. Frolov, *Foundations of the $AdS_5 \times S^5$ superstring. part I*, arXiv:0901.4937 [SPIRES].

[12] N. Drukker, D.J. Gross and A.A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function*, JHEP **04** (2000) 021 [hep-th/0001204] [SPIRES].

[13] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, JHEP **06** (2002) 007 [hep-th/0204226] [SPIRES]; *Quantizing three-spin string solution in $AdS_5 \times S^5$*, JHEP **07** (2003) 016 [hep-th/0306130] [SPIRES];
I.Y. Park, A. Tirziu and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in* SL(2) *sector*, JHEP **03** (2005) 013 [hep-th/0501203] [SPIRES];
S. Frolov, A. Tirziu and A.A. Tseytlin, *Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT*, Nucl. Phys. **B 766** (2007) 232 [hep-th/0611269] [SPIRES];
N. Gromov and P. Vieira, *The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve*, Nucl. Phys. **B 789** (2008) 175 [hep-th/0703191] [SPIRES].

[14] R. Roiban, A. Tirziu and A.A. Tseytlin, *Two-loop world-sheet corrections in $AdS_5 \times S^5$ superstring*, JHEP **07** (2007) 056 [arXiv:0704.3638] [SPIRES];
R. Roiban and A.A. Tseytlin, *Strong-coupling expansion of cusp anomaly from quantum superstring*, JHEP **11** (2007) 016 [arXiv:0709.0681] [SPIRES].

[15] N. Beisert, A.A. Tseytlin and K. Zarembo, *Matching quantum strings to quantum spins: one-loop vs. finite-size corrections*, Nucl. Phys. **B 715** (2005) 190 [hep-th/0502173] [SPIRES];
S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study*, JHEP **09** (2005) 051 [hep-th/0507189] [SPIRES];
S. Schäfer-Nameki and M. Zamaklar, *Stringy sums and corrections to the quantum string Bethe ansatz*, JHEP **10** (2005) 044 [hep-th/0509096] [SPIRES];
P.Y. Casteill and C. Kristjansen, *The strong coupling limit of the scaling function from the quantum string Bethe ansatz*, Nucl. Phys. **B 785** (2007) 1 [arXiv:0705.0890] [SPIRES];
N. Beisert and A.A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, Phys. Lett. **B 629** (2005) 102 [hep-th/0509084] [SPIRES];
R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, JHEP **07** (2006) 004 [hep-th/0603204] [SPIRES].

[16] M.K. Benna, S. Benvenuti, I.R. Klebanov and A. Scardicchio, *A test of the AdS/CFT correspondence using high-spin operators*, Phys. Rev. Lett. **98** (2007) 131603 [hep-th/0611135] [SPIRES];
B. Basso, G.P. Korchemsky and J. Kotanski, *Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling*, Phys. Rev. Lett. **100** (2008) 091601 [arXiv:0708.3933] [SPIRES];
N. Gromov, *Generalized scaling function at strong coupling*, JHEP **11** (2008) 085 [arXiv:0805.4615] [SPIRES].

[17] N. Berkovits, *Quantum consistency of the superstring in $AdS_5 \times S^5$ background*, JHEP **03** (2005) 041 [hep-th/0411170] [SPIRES];
B.C. Vallilo, *Flat currents in the classical $AdS_5 \times S^5$ pure spinor superstring*, JHEP **03** (2004) 037 [hep-th/0307018] [SPIRES].

[18] A. Mikhailov and S. Schäfer-Nameki, *Algebra of transfer-matrices and Yang-Baxter equations on the string worldsheet in $AdS_5 \times S^5$*, Nucl. Phys. **B 802** (2008) 1 [arXiv:0712.4278] [SPIRES].

[19] V.G.M. Puletti, *Aspects of quantum integrability for pure spinor superstring in $AdS_5 \times S_5$*, JHEP **09** (2008) 070 [arXiv:0808.0282] [SPIRES].

[20] C.G. Callan Jr. et al., *Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave*, Nucl. Phys. **B 673** (2003) 3 [hep-th/0307032] [SPIRES];
C.G. Callan Jr., T. McLoughlin and I. Swanson, *Holography beyond the Penrose limit*, Nucl. Phys. **B 694** (2004) 115 [hep-th/0404007] [SPIRES].

[21] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, *The off-shell symmetry algebra of the light-cone $AdS_5 \times S^5$ superstring*, J. Phys. **A 40** (2007) 3583 [hep-th/0609157] [SPIRES];
G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$*, JHEP **02** (2005) 059 [hep-th/0411089] [SPIRES].

[22] A.M. Polyakov, *Conformal fixed points of unidentified gauge theories*, Mod. Phys. Lett. **A 19** (2004) 1649 [hep-th/0405106] [SPIRES].

[23] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. **B 625** (2002) 70 [hep-th/0112044] [SPIRES];
R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, Phys. Rev. **D 65** (2002) 126004 [hep-th/0202109] [SPIRES].

[24] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, JHEP **04** (2002) 013 [hep-th/0202021] [SPIRES].

[25] J.L. Miramontes, *Pohlmeyer reduction revisited*, JHEP **10** (2008) 087 [arXiv:0808.3365] [SPIRES].

[26] N. Beisert, *The* SU(2|2) *dynamic S-matrix*, Adv. Theor. Math. Phys. **12** (2008) 945 [hep-th/0511082] [SPIRES].

[27] D.M. Hofman and J.M. Maldacena, *Giant magnons*, J. Phys. **A 39** (2006) 13095 [hep-th/0604135] [SPIRES];
H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, JHEP **09** (2006) 024 [hep-th/0605155] [SPIRES];
A. Jevicki and K. Jin, *Solitons and AdS string solutions*, Int. J. Mod. Phys. **A 23** (2008) 2289 [arXiv:0804.0412] [SPIRES];
A. Jevicki, K. Jin, C. Kalousios and A. Volovich, *Generating AdS string solutions*, JHEP **03** (2008) 032 [arXiv:0712.1193] [SPIRES];

T. Klose and T. McLoughlin, *Interacting finite-size magnons*, J. Phys. **A 41** (2008) 285401 [arXiv:0803.2324] [SPIRES].

[28] L.F. Alday, G. Arutyunov and A.A. Tseytlin, *On integrability of classical superstrings in $AdS_5 \times S^5$*, JHEP **07** (2005) 002 [hep-th/0502240] [SPIRES];
L.F. Alday, G. Arutyunov and S. Frolov, *New integrable system of 2dim fermions from strings on $AdS_5 \times S^5$*, JHEP **01** (2006) 078 [hep-th/0508140] [SPIRES];
G. Arutyunov, *Lecture notes on $AdS_5 \times S^5$ superstring*, unpublished (2006).

[29] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *Superstring theory on $AdS_2 \times S^2$ as a coset supermanifold*, Nucl. Phys. **B 567** (2000) 61 [hep-th/9907200] [SPIRES].

[30] H.J. Schnitzer, *A path integral construction of superconformal field theories from a gauged supersymmetric Wess-Zumino-Witten action*, Nucl. Phys. **B 324** (1989) 412 [SPIRES];
A.A. Tseytlin, *Conformal σ-models corresponding to gauged Wess-Zumino-Witten theories*, Nucl. Phys. **B 411** (1994) 509 [hep-th/9302083] [SPIRES].

[31] R. Roiban and W. Siegel, *Superstrings on $AdS_5 \times S^5$ supertwistor space*, JHEP **11** (2000) 024 [hep-th/0010104] [SPIRES].

[32] I. Bakas, Q.-H. Park and H.-J. Shin, *Lagrangian formulation of symmetric space Sine-Gordon models*, Phys. Lett. **B 372** (1996) 45 [hep-th/9512030] [SPIRES].

[33] I. Bars and K. Sfetsos, *A superstring theory in four curved space-time dimensions*, Phys. Lett. **B 277** (1992) 269 [hep-th/9111040] [SPIRES]; *Global analysis of new gravitational singularities in string and particle theories*, Phys. Rev. **D 46** (1992) 4495 [hep-th/9205037] [SPIRES].

[34] E.S. Fradkin and V.Y. Linetsky, *On space-time interpretation of the coset models in $D < 26$ critical string theory*, Phys. Lett. **B 277** (1992) 73 [SPIRES].

[35] D.H. Friedan, *Nonlinear models in two + epsilon dimensions*, Ann. Phys. **163** (1985) 318 [SPIRES];
C.G. Callan Jr. and Z. Gan, *Vertex operators in background fields*, Nucl. Phys. **B 272** (1986) 647 [SPIRES].

[36] A.A. Tseytlin, *Conformal anomaly in two-dimensional σ-model on curved background and strings*, Phys. Lett. **178B** (1986) 34 [SPIRES].

[37] H. Osborn, *General bosonic σ-models and string effective actions*, Ann. Phys. **200** (1990) 1 [SPIRES].

[38] A.A. Tseytlin, *On field redefinitions and exact solutions in string theory*, Phys. Lett. **B 317** (1993) 559 [hep-th/9308042] [SPIRES].

[39] R.R. Metsaev and A.A. Tseytlin, *Order α' (two loop) equivalence of the string equations of motion and the σ-model Weyl invariance conditions: dependence on the dilaton and the antisymmetric tensor*, Nucl. Phys. **B 293** (1987) 385 [SPIRES].

[40] M. Bos, *An example of dimensional regularization with antisymmetric tensors*, Ann. Phys. **181** (1988) 177 [SPIRES].

[41] B. de Wit, M.T. Grisaru and P. van Nieuwenhuizen, *The WZNW model at two loops*, Nucl. Phys. **B 408** (1993) 299 [hep-th/9307027] [SPIRES].

[42] B.E. Fridling and A.E.M. van de Ven, *Renormalization of generalized two-dimensional nonlinear σ-models*, Nucl. Phys. **B 268** (1986) 719 [SPIRES];

JHEP04(2009)078

C.M. Hull and P.K. Townsend, *The two loop β-function for σ-models with torsion*, Phys. Lett. **B 191** (1987) 115 [SPIRES].

[43] V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. **B 247** (1984) 83 [SPIRES].

[44] I. Jack and D.R.T. Jones, *The exact tachyon β-function for the Wess-Zumino-Witten model*, Mod. Phys. Lett. **A 9** (1994) 259 [hep-th/9310029] [SPIRES].

[45] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *String propagation in a black hole geometry*, Nucl. Phys. **B 371** (1992) 269 [SPIRES].

[46] A.A. Tseytlin, *On the form of the black hole solution in $D = 2$ theory*, Phys. Lett. **B 268** (1991) 175 [SPIRES].

[47] A.A. Tseytlin, *Effective action of gauged WZW model and exact string solutions*, Nucl. Phys. **B 399** (1993) 601 [hep-th/9301015] [SPIRES];
K. Sfetsos and A.A. Tseytlin, *Antisymmetric tensor coupling and conformal invariance in σ-models corresponding to gauged WZNW theories*, Phys. Rev. **D 49** (1994) 2933 [hep-th/9310159] [SPIRES].

[48] A.N. Redlich and H.J. Schnitzer, *The Polyakov string in $O(N)$ or $\mathrm{SU}(N)$ group space*, Phys. Lett. **B 167** (1986) 315 [*Erratum ibid.* **B 193** (1987) 536] [SPIRES];
J. Fuchs, *More on the super WZW theory*, Nucl. Phys. **B 318** (1989) 631 [SPIRES].

[49] A.M. Polyakov and P.B. Wiegmann, *Theory of nonabelian Goldstone bosons in two dimensions*, Phys. Lett. **B 131** (1983) 121 [SPIRES]; *Goldstone fields in two-dimensions with multivalued actions*, Phys. Lett. **B 141** (1984) 223 [SPIRES];
P. Di Vecchia and P. Rossi, *On the equivalence between the Wess-Zumino action and the free Fermi theory in two dimensions*, Phys. Lett. **B 140** (1984) 344 [SPIRES];
P. Di Vecchia, B. Durhuus and J.L. Petersen, *The Wess-Zumino action in two dimensions and nonabelian bosonization*, Phys. Lett. **B 144** (1984) 245 [SPIRES].

[50] H. Aratyn, J.F. Gomes and A.H. Zimerman, *Supersymmetry and the KdV equations for integrable hierarchies with a half-integer gradation*, Nucl. Phys. **B 676** (2004) 537 [hep-th/0309099] [SPIRES];
J.F. Gomes, D.M. Schmidtt and A.H. Zimerman, *Super WZNW with reductions to supersymmetric and fermionic integrable models*, arXiv:0901.4040 [SPIRES].

[51] W. Siegel, *Supersymmetric dimensional regularization via dimensional reduction*, Phys. Lett. **B 84** (1979) 193 [SPIRES].

[52] S.J. Gates Jr., M.T. Grisaru, L. Mezincescu and P.K. Townsend, $(1,0)$ *supergraphity*, Nucl. Phys. **B 286** (1987) 1 [SPIRES].

[53] N. Beisert, R. Ricci, A.A. Tseytlin and M. Wolf, *Dual superconformal symmetry from $AdS_5 \times S^5$ superstring integrability*, Phys. Rev. **D 78** (2008) 126004 [arXiv:0807.3228] [SPIRES].

[54] I. Jack, D.R.T. Jones and J. Panvel, *Quantum non abelian Toda field theories*, Int. J. Mod. Phys. **A 9** (1994) 3631 [hep-th/9308080] [SPIRES].

[55] M.T. Grisaru, D.I. Kazakov and D. Zanon, *Five loop divergences for the $N = 2$ supersymmetric nonlinear σ-model*, Nucl. Phys. **B 287** (1987) 189 [SPIRES].

[56] M. Blau, J.M. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, Class. Quant. Grav. **19** (2002) L87 [hep-th/0201081] [SPIRES].

[57] U. Ellwanger, J. Fuchs and M.G. Schmidt, *The heterotic σ-model with background gauge fields*, *Nucl. Phys.* **B 314** (1989) 175 [SPIRES].

[58] C.M. Hull, G. Papadopoulos and P.K. Townsend, *Potentials for* $(p, 0)$ *and* $(1, 1)$ *supersymmetric σ-models with torsion*, *Phys. Lett.* **B 316** (1993) 291 [hep-th/9307013] [SPIRES];
G. Papadopoulos and P.K. Townsend, *Massive σ-models with (p,q) supersymmetry*, *Class. Quant. Grav.* **11** (1994) 515 [hep-th/9307066] [SPIRES].

[59] N.D. Lambert, *Two loop renormalization of massive (p,q) supersymmetric σ-models*, *Nucl. Phys.* **B 469** (1996) 68 [hep-th/9510130] [SPIRES].

[60] R.W. Allen and D.R.T. Jones, *The $N = 1$ supersymmetric σ-model with torsion: the two loop β-functions*, *Nucl. Phys.* **B 303** (1988) 271 [SPIRES].

[61] P. Di Vecchia, V.G. Knizhnik, J.L. Petersen and P. Rossi, *A supersymmetric Wess-Zumino Lagrangian in two-dimensions*, *Nucl. Phys.* **B 253** (1985) 701 [SPIRES];
E. Abdalla and M.C.B. Abdalla, *Supersymmetric extension of the chiral model and Wess-Zumino term in two-dimensions*, *Phys. Lett.* **B 152** (1985) 59 [SPIRES].

[62] O.A. Castro Alvaredo and J.L. Miramontes, *Massive symmetric space sine-Gordon soliton theories and perturbed conformal field theory*, *Nucl. Phys.* **B 581** (2000) 643 [hep-th/0002219] [SPIRES].

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